# SUPPLEMENT TO
# CALCULUS

Michael Spivak

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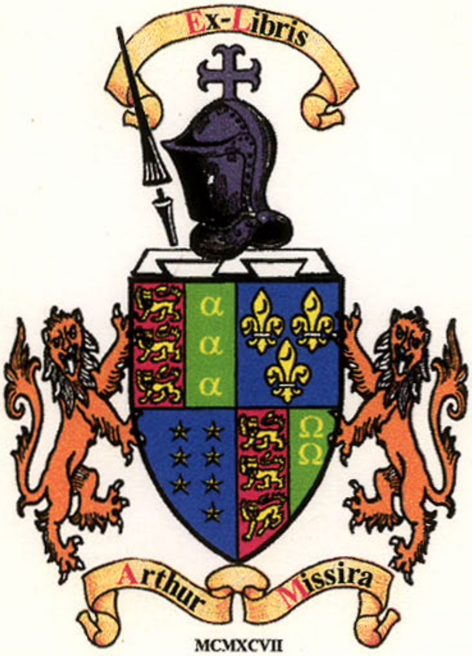

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# SUPPLEMENT

# TO

# CALCULUS

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SUPPLEMENT TO CALCULUS

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This book contains solutions to all problems in "Calculus," except for those which are solved in the back of the text itself. The answers were written during a (very unpleasant) two-month period and there are bound to be, aside from a generous sprinkling of misprints, errors of a more substantial kind. Computational errors, at least, should not be hard to spot, since most computations are worked out in considerable detail. Greater difficulties will be encountered in the case of proofs, especially since they have been presented with a conciseness somewhat at variance with the style of the text. Considerations of space alone did not suggest this--answer books should always be used only as a last resort, and it is often easier to construct a proof on one's own than to read someone else's. In some of the later chapters a few mistakes in the original problems were discovered, too late to be rectified, as these answers were being written; the solutions will, I trust, reveal any mistakes which might escape the reader.

M.S.

# CHAPTER 1

1. (ii)
$$\begin{aligned}(x - y)(x + y) &= [x + (-y)](x + y) = x(x + y) + (-y)(x + y)\\ &= x(x + y) - [y(x + y)] = x^2 + xy - [yx + y^2]\\ &= x^2 + xy - xy - y^2 = x^2 - y^2.\end{aligned}$$

(iv)
$$\begin{aligned}(x - y)(x^2 + xy + y^2) &= x(x^2 + xy + y^2) - [y(x^2 + xy + y^2)]\\ &= x^3 + x^2y + xy^2 - [yx^2 + xy^2 + y^3] = x^3 - y^3.\end{aligned}$$

(v)
$$\begin{aligned}&(x - y)(x^{n-1} + x^{n-2}y + \dots + xy^{n-2} + y^{n-1})\\ &= x(x^{n-1} + x^{n-2}y + \dots + xy^{n-2} + y^{n-1})\\ &\qquad - [y(x^{n-1} + x^{n-2}y + \dots + xy^{n-2} + y^{n-1})]\\ &= x^n + x^{n-1}y + \dots + x^2y^{n-2} + xy^{n-1}\\ &\qquad - [x^{n-1}y + x^{n-2}y^2 + \dots + xy^{n-1} + y^n] = x^n - y^n.\end{aligned}$$

(Using the notation of Chapter 2, this proof can be written as follows:

$$(x - y) \cdot \sum_{j=0}^{n-1} x^j y^{n-1-j} = x(\sum_{j=0}^{n-1} x^j y^{n-1-j}) - [y(\sum_{j=0}^{n-1} x^j y^{n-1-j})]$$

$$= x^n + \sum_{j=0}^{n-2} x^{j+1} y^{n-1-j} - [\sum_{j=1}^{n-1} x^j y^{n-j} + y^n]$$

$$= x^n + \sum_{j=0}^{n-2} x^{j+1} y^{n-1-j} - [\sum_{k=0}^{n-2} x^{k+1} y^{n-(k+1)} + y^n]$$

(letting $k = j - 1$)

$$= x^n - y^n.$$

A formal proof requires such a scheme, in which the expression $(x^{n-1} + x^{n-2}y + \ldots + xy^{n-2} + y^{n-1})$ is replaced by the inductively defined symbol $\sum_{j=0}^{n-1} x^j y^{n-1-j}$. Along the way we have used several other manipulations which can, if necessary, be justified by inductive arguments.)

3. (iv) $(a/b)(c/d) = (ab^{-1})(cd^{-1}) = (ac)(b^{-1}d^{-1}) = (ac)(bd)^{-1}$ (by (iii)) $= (ac)/(bd)$.

4. (ii) All x.
   (iv) $x > 3$ or $x < 1$.
   (vi) $x > [-1 + \sqrt{5}]/2$ or $x < [-1 - \sqrt{5}]/2$.
   (viii) All x, since $x^2 + x + 1 = [x + (1/2)]^2 + 3/4$.
   (x) $x > \sqrt{2}$ or $x < \sqrt[3]{2}$.
   (xii) $x < 1$.
   (xiv) $x > 1$ or $x < -1$.

5. (ii) $b - a$ is in P, so $-a - (-b)$ is in P.
   (iv) $b - a$ is in P and c is in P, so $c(b - a) = bc - ac$ is in P.
   (vi) If $a > 1$, then $a > 0$, so $a^2 > a \cdot 1$, by part (iv).
   (viii) If $a = 0$ or $c = 0$, then $ac = 0$, but $bd > 0$, so $ac < bd$. Otherwise we have $ac < bc < bd$ by applying part (iv) twice.

(x) If $a < b$ were false, then either $a = b$ or $a > b$. But if $a = b$, then $a^2 = b^2$, and if $a > b \geq 0$, then $a^2 > b^2$, by part (ix).

6. If $a < b$, then

$$a = \frac{a+a}{2} < \frac{a+b}{2} < \frac{b+b}{2} = b.$$

If $0 < a < b$, then $a^2 < ab$ by Problem 5(iv), so $a < \sqrt{ab}$ by Problem 5(x). Moreover, $(a - b)^2 > 0$, so

$$a^2 + b^2 > 2ab,$$

$$a^2 + 2ab + b^2 > 4ab,$$

$$(a + b)^2 > 4ab,$$

so $a + b > 2\sqrt{ab}$.

7. (a) If $x > 0$ and $y \leq 0$, then $x^n > 0$ and $y^n \leq 0$ (since $n$ is odd), and similarly if $x \leq 0$ and $y > 0$; so neither alternative holds if $x^n = y^n$. Moreover, if $x^n = y^n$ and $x, y < 0$, then $(-x)^n = (-y)^n$; so if the assertion is true for positive numbers, then $-x = -y$, and consequently $x = y$. Thus it suffices to consider only $x, y > 0$. In this case, $x < y$ implies $x^n < y^n$, and similarly for $y < x$, so neither alternative can hold.

(b) If $x, y > 0$ or $x, y < 0$, then $x = y$, as in part (a). If $x < 0$ and $y > 0$, then $(-x)^n = y^n$, since $n$ is even, so $-x = y$; similarly if $x > 0$ and $y < 0$.

8. Two applications of P'12 show that if $a < b$ and $c < d$, then $a + c < b + c < b + d$, so $a + c < b + d$ by P'11. In particular, if $0 < b$ and $0 < d$, then $0 < b + d$, which proves P11. It follows, in addition, that if $a < 0$, then $-a > 0$; for if $-a < 0$ were true, then $0 = a + (-a) < 0$, contradicting P'10. Consequently, any number a satisfies precisely one of the conditions $a = 0$, $a > 0$, $a < 0$, the last being equivalent to $-a > 0$. This proves P10. Finally, P'13 shows that if $0 < a$ and $0 < c$, then $0 < ac$, which proves P12.

9. (ii) $|a| + |b| - |a + b|$.

(iv) $x^2 - 2xy + y^2$.

10. (ii) $x - 1$ if $x \geq 1$;

$1 - x$ if $0 \leq x \leq 1$;

$1 + x$ if $-1 \leq x \leq 0$;

$-1 - x$ if $x \leq -1$.

(iv) $a$ if $a \geq 0$;

$3a$ if $a \leq 0$.

11. (ii) $-5 < x < 11$.

(iv) $x < 1$ or $x > 2$ (the distance from x to 1 plus the distance from x to 2 equals 1 precisely when $1 \leq x \leq 2$).

(vi) No x.

(viii) If $x > 1$ or $x < -2$, then the condition becomes $(x - 1)(x - 2) = 3$, or $x^2 + x - 5 = 0$, for which the solutions are $(-1 + \sqrt{21})/2$ and $(-1 - \sqrt{21})/2$. Since the first is $> 1$ and the second is $< -2$, both are solutions to $|x - 1| \cdot |x + 2| = 3$. For $-2 < x < 1$ the condition becomes $(1 - x)(x - 2) = 3$ or $x^2 + x + 1 = 0$, which has no solutions.

12. (ii) $|1/x| \cdot |x| = |(1/x) \cdot x|$ (by (i)) $= |1| = 1$, so $|1/x| = 1/|x|$.

(iv) $|x - y| = |x + (-y)| \leq |x| + |-y| = |x| + |y|$.

(vi) Interchanging x and y in part (v) gives $|y| - |x| \leq |x - y|$. Combining this with part (v) yields $|(|x| - |y|)| \leq |x - y|$.

13. If $x \leq y$, then $|y - x| = y - x$, so $x + y + |y - x| = x + y + y - x = 2y$, which is $2 \max(x, y)$. Interchanging x and y proves the formula when $x \geq y$, and the same type of argument works for $\min(x, y)$. Also,

$$\max(x, y, z) = \max(x, \max(y, z)) = \frac{x + \dfrac{y + z + |y - z|}{2} + \left|\dfrac{y + z + |y - z|}{2} - x\right|}{2}$$

$$= \frac{|y - z| + y + z + 2x + |y + z + |y - z| - 2x|}{4}.$$

14. (a) If $a \geq 0$, then $|a| = a = -(-a) = |-a|$, since $-a \leq 0$. The equality is proved for $a \leq 0$ by replacing a by $-a$.

(b) If $|a| \leq b$, then clearly $b \geq 0$. Now $|a| \leq b$ means that $a \leq b$ if $a \geq 0$, and surely $a \leq b$ if $a \leq 0$. Similarly, $|a| \leq b$ means $-a \leq b$, and hence $-b \leq a$, if $a \leq 0$, and surely $-b \leq a$ if $a \geq 0$. So $-b \leq a \leq b$.

Conversely, if $-b \leq a \leq b$, then $|a| = a \leq b$ if $a \geq 0$, while $|a| = -a \leq b$ if $a \leq 0$.

(c) From $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$ it follows that

$$-(|a| + |b|) \leq a + b \leq |a| + |b|,$$

so $|a + b| \leq |a| + |b|$.

15. (a) If $x^2 + xy + y^2 = 0$, then $x^3 - y^3 = (x - y)(x^2 + xy + y^2) = 0$, so $x = y$. But in this case, $x^2 + xy + y^2 = 3x^2 \neq 0$, since $x \neq 0$, a contradiction. The other inequality is proved similarly.

(b) From the known, and the assumed, inequalities

$$x^2 + 2xy + y^2 \geq 0,$$
$$x^2 + xy + y^2 < 0,$$

it follows, by subtracting, that $xy \geq 0$. But this implies that $x^2 + xy + y^2 \geq 0$, contradicting the original assumption.

(c) From

$$4x^2 + 8xy + 4y^2 \geq 0,$$
$$4x^2 + 6xy + 4y^2 < 0,$$

it would follow that $2xy \geq 0$, contradicting the assumed inequality, as in part (a). The other inequality is handled similarly.

(d) From

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 \geq 0,$$
$$x^4 + x^3y + x^2y^2 + xy^3 + y^4 < 0,$$

it follows that

$$0 < 3x^3y + 5x^2y^2 + 3xy^3 = xy(3x^2 + 5xy + 3y^2).$$

If x and y are not both 0, then the expression in parentheses is positive, by part (c), so $xy \geq 0$, so x and y have the same sign. But this implies that

$$x^4 + x^3y + x^2y^2 + xy^3 + y^4 \geq 0,$$

a contradiction.

16. (a) If

$$x^2 + y^2 = (x + y)^2 = x^2 + 2xy + y^2,$$

then $xy = 0$, so $x = 0$ or $y = 0$. If

$$x^3 + y^3 = (x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3,$$

then $3xy(x + y) = 0$, so $x = 0$ or $y = 0$ or $x = -y$.

(b) If

$$x^4 + y^4 = (x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4,$$

then

$$0 = 4x^3y + 6x^2y^2 + 4xy^3 = xy(4x^2 + 6xy + 4y^2),$$

so $x = 0$ or $y = 0$ by Problem 15(c).

(c) If

$$\begin{aligned} x^5 + y^5 &= (x + y)^5 \\ &= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5, \end{aligned}$$

then

$$\begin{aligned} 0 &= 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 \\ &= 5xy(x^3 + 2x^2y + 2xy^2 + y^3), \end{aligned}$$

so $xy = 0$ or

$$x^3 + 2x^2y + 2xy^2 + y^3 = 0.$$

Subtracting this equation from

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

we obtain

$$(x + y)^3 = x^2y + xy^2 = xy(x + y).$$

So either $x + y = 0$ or $(x + y)^2 = xy$; the latter condition implies that $x^2 + xy + y^2 = 0$, so $x = 0$ or $y = 0$ by Problem 15(a). Thus $x = 0$ or $y = 0$ or $x = -y$.

17. (a) is a straightforward check.

(b) We have

$$x^2 + bx + c = (x + \frac{b}{2})^2 + (c - \frac{b^2}{4}) \geq c - \frac{b^2}{4};$$

but $c - b^2/4 > 0$, so $x^2 + bx + c > 0$ for all x.

(c) Apply part (b) with y for b and $y^2$ for c: we have $b^2 - 4c = y^2 - 4y^2 < 0$ for $y \neq 0$, so $x^2 + xy + y^2 > 0$ for all x, if $y \neq 0$ (and surely $x^2 + xy + y^2 > 0$ for all $x \neq 0$ if $y = 0$).

(d) $\alpha$ must satisfy $(\alpha y)^2 - 4y^2 < 0$, or $\alpha^2 < 4$, or $|\alpha| < 2$.

(e) Since

$$x^2 + bx + c = (x + \frac{b}{2})^2 + (c - \frac{b^2}{4}) \geq c - \frac{b^2}{4},$$

and since $x^2 + bx + c$ has the value $c - b^2/4$ when $x = -b/2$, the minimum value is $c - b^2/4$. Since

$$ax^2 + bx + c = a(x^2 + \frac{b}{a}x + \frac{c}{a}),$$

the minimum value is

$$a(\frac{c}{a} - \frac{b^2}{4a^2}) = c - \frac{b^2}{4a}.$$

18. (a) The equality is a straightforward computation. Since $(x_1y_2 - x_2y_1)^2 \geq 0$, the Schwartz inequality follows immediately.

(b) The proofs when $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$, or $y_1 = y_2 = 0$, are straightforward. If there is no such $\lambda$, then the equation

$$\lambda^2(y_1^2 + y_2^2) - 2\lambda(x_1y_1 + x_2y_2) + (x_1^2 + y_1^2) = 0$$

has no solution $\lambda$, so by Problem 17(a) we have

$$\left[\frac{2(x_1y_1 + x_2y_2)}{(y_1^2 + y_2^2)}\right]^2 - \frac{4(x_1^2 + y_1^2)}{(y_1^2 + y_2^2)} < 0,$$

which yields the Schwartz inequality.

(c) We have $2xy \leq x^2 + y^2$, since $0 \leq (x - y)^2 = x^2 - 2xy + y^2$. Thus

$$(1) \qquad \frac{2x_1y_1}{\sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}} \leq \frac{x_1^2}{(x_1^2 + x_2^2)} + \frac{y_1^2}{(y_1^2 + y_2^2)},$$

$$(2) \qquad \frac{2x_2y_2}{\sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}} \leq \frac{x_2^2}{(x_1^2 + x_2^2)} + \frac{y_2^2}{(y_1^2 + y_2^2)};$$

addition yields

$$\frac{2(x_1y_1 + x_2y_2)}{\sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}} \leq 2.$$

(d) In part (a), equality holds only when $x_1y_2 - x_2y_1 = 0$. One possibility is $y_1 = y_2 = 0$. If $y_1 \neq 0$, then $x_1 = (x_1/y_1)y_1$ and also $x_2 = (x_1/y_1)y_2$; similarly, if $y_2 \neq 0$, then $\lambda = x_2/y_2$.

The proof in part (b) already yields the desired result.

In part (c), equality holds only if it holds in (1) and (2). Since $2xy = x^2 + y^2$ only when $0 = (x - y)^2$, i.e., $x = y$, this means that

$$\frac{x_i}{\sqrt{x_1^2 + x_2^2}} = \frac{y_i}{\sqrt{y_1^2 + y_2^2}} \qquad \text{for } i = 1, 2,$$

so we can choose $\lambda = \sqrt{x_1^2 + x_2^2}/\sqrt{y_1^2 + y_2^2}$.

19, 20, 21. See Chapter 5.

22. According to Problem 20, we have $|x/y - x_0/y_0| < \epsilon$ if

$$|x - x_0| < \min\left(\frac{\epsilon}{2(1/|y_0| + 1)}, 1\right)$$

and

$$\left|\frac{1}{y} - \frac{1}{y_0}\right| < \frac{\epsilon}{2(|x_0| + 1)},$$

and the latter is true, according to Problem 21, if

$$|y - y_0| < \min\left(\frac{|y_0|}{2}, \frac{\epsilon |y_0|^2}{4(|x_0| + 1)}\right).$$

23. (a) For $k = 1$ the equation reads $a_1 + a_2 - a_1 + a_2$. If the equation holds for $k$, then

$$\begin{aligned}
(a_1 + \dots + a_{k+1}) + a_{k+2} &= [(a_1 + \dots + a_k) + a_{k+1}] + a_{k+2} \\
&= (a_1 + \dots + a_k) + (a_{k+1} + a_{k+2}) && \text{by P1} \\
&= a_1 + \dots + a_k + (a_{k+1} + a_{k+2}) && \text{since the equation holds for } k \\
&= a_1 + \dots + a_{k+2} && \text{by the definition of } a_1 + \dots + a_{k+2}.
\end{aligned}$$

(b) For $k = 1$ the equation reduces to the definition of $a_1 + \dots + a_k$. If the equation is true for some $k < n$, then

$$\begin{aligned}
(a_1 + \dots + a_{k+1}) + (a_{k+2} + \dots + a_n) &= ([a_1 + \dots + a_k] + a_{k+1}) \\
&\qquad + (a_{k+2} + \dots + a_n) && \text{by part (a)} \\
&= (a_1 + \dots + a_k) \\
&\qquad + (a_{k+1} + (a_{k+2} + \dots + a_n)) && \text{by P1} \\
&= (a_1 + \dots + a_k) + (a_{k+1} + \dots + a_n) \\
&\qquad \text{by the definition of } a_{k+1} + \dots + a_n \\
&= a_1 + \dots + a_n && \text{by assumption.}
\end{aligned}$$

(c) The proof is by "complete induction" on k (see Chapter 2). The assertion is clear for $k = 1$. Assume that is true for all $\ell < k$. Then

$$\begin{aligned} s(a_1,\dots,a_k) &= s'(a_1,\dots,a_\ell) + s''(a_{\ell+1},\dots,a_k) \\ &= (a_1 + \dots + a_\ell) + (a_{\ell+1} + \dots + a_k) \qquad \text{by assumption} \\ &= a_1 + \dots + a_k \qquad \text{by part (b).} \end{aligned}$$

24. P2, P3, P4, P6, P7, P8 are obvious from a glance at the tables. There are eight cases for P1, and even this number can be reduced: because P2 is true, it is clear that $a + (b + c) = (a + b) + c$ if a, b, or c is 0, so only the case $a = b = c = 1$ must be checked. Similarly for P5. Finally, P9 is true for $a = 0$, since $0 \cdot b = 0$ for all b, and for $a = 1$, since $1 \cdot b = b$ for all b.

# CHAPTER 2

1. (ii) Since $1^3 = 1^2$, the formula is true for $n = 1$. Suppose that the formula is true for $k$. Then

$$\begin{aligned}(1 + \dots + k + [k+1])^2 &= (1 + \dots + k)^2 + 2(1 + \dots + k)(k+1) + (k+1)^2 \\ &= 1^3 + \dots + k^3 + 2\,\frac{k(k+1)}{2}(k+1) + (k+1)^2 \\ &= 1^3 + \dots + k^3 + (k^3 + 2k^2 + k) + (k^2 + 2k + 1) \\ &= 1^3 + \dots + k^3 + (k+1)^3,\end{aligned}$$

so the formula is true for $k + 1$.

2. (ii)

$$\begin{aligned}\sum_{i=1}^{n} (2i-1)^2 &= 1^2 + 3^2 + \dots + (2n-1)^2 \\ &= [1^2 + 2^2 + \dots + (2n)^2] - [2^2 + 4^2 + 6^2 + \dots + (2n)^2] \\ &= [1^2 + 2^2 + \dots + (2n)^2] - 4[1^2 + 2^2 + 3^2 + \dots + n^2] \\ &= \frac{2n(2n+1)(4n+1)}{6} - \frac{4n(n+1)(2n+1)}{6} \\ &= \frac{2n(2n+1)[4n+1-2(n+1)]}{6} \\ &= \frac{n(2n+1)(2n-1)}{3}.\end{aligned}$$

3. (a)

$$\begin{aligned}\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} \\ &= \frac{kn!}{k!(n+1-k)!} + \frac{(n+1-k)n!}{k!(n+1-k)!} \\ &= \frac{(n+1)n!}{k!(n+1-k)!} = \binom{n+1}{k}.\end{aligned}$$

(b) Clearly $\binom{1}{1}$ is a natural number. Suppose that $\binom{n}{p}$ is a natural number for all $p \leq n$. Since

$$\binom{n+1}{p} = \binom{n}{p-1} + \binom{n}{p} \qquad \text{for } p \leq n,$$

it follows that $\binom{n+1}{p}$ is a natural number for all $p \leq n$, while $\binom{n+1}{n+1}$ is also a natural number. So $\binom{n+1}{p}$ is a natural number for all $p \leq n + 1$.

(c) There are $n(n-1)\cdot\ldots\cdot(n-k+1)$ k-tuples of distinct integers each chosen from $1,\ldots,n$, since the first can be picked in n ways, the next in $n - 1$ ways, etc. Now each set of exactly k integers can be arranged in $k!$ k-tuples, so there are $n(n-1)\cdot\ldots\cdot(n-k+1)/k! = \binom{n}{k}$ such sets.

(d) The binomial theorem is clear for $n = 1$. Suppose that

$$(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^j.$$

Then

$$\begin{aligned}(a+b)^{n+1} &= (a+b)(a+b)^n = (a+b) \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^j \\ &= \sum_{j=0}^{n} \binom{n}{j} a^{n+1-j} b^j + \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^{j+1} \\ &= \sum_{j=0}^{n} \binom{n}{j} a^{n+1-j} b^j + \sum_{j=1}^{n+1} \binom{n}{j-1} a^{n+1-j} b^j\end{aligned}$$

(we have replaced j by $j - 1$ in the second sum)

$$= \sum_{j=0}^{n+1} \binom{n+1}{j} a^{n+1-j} b^j \qquad \text{by part (a),}$$

so the binomial theorem is true for $n + 1$.

(e) (i) $$2^n = (1 + 1)^n = \sum_{j=0}^{n} \binom{n}{j},$$

(ii) $$0 = (1 + -1)^n = \sum_{j=0}^{n} (-1)^j \binom{n}{j}.$$

5. (ii) From

$$(k + 1)^5 - k^5 = 5k^4 + 10k^3 + 10k^2 + 5k + 1 \qquad k = 1,\ldots,n$$

we obtain

$$(n + 1)^5 - 1 = 5\left(\sum_{k=1}^{n} k^4\right) + 10\left(\sum_{k=1}^{n} k^3\right) + 10\left(\sum_{k=1}^{n} k^2\right) + 5\left(\sum_{k=1}^{n} k\right) + n,$$

so

$$\sum_{k=1}^{n} k^4 = \frac{(n + 1)^5 - 1 - 10\left(\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}\right) - 10\,\frac{n(n + 1)(n + 2)}{6} - 5\,\frac{n(n + 1)}{2} - n}{5}$$

$$= \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}.$$

(iv) From

$$\frac{1}{k^2} - \frac{1}{(k + 1)^2} = \frac{2k + 1}{k^2(k + 1)^2} \qquad k = 1,\ldots,n$$

we obtain

$$1 - \frac{1}{(n + 1)^2} = \sum_{k=1}^{n} \frac{2k + 1}{k^2(k + 1)^2}.$$

6. The proof is by complete induction on p. The statement is true for p = 1, since

$$\sum_{k=1}^{n} k = \frac{n(n + 1)}{2} = \frac{n^2}{2} + n.$$

Suppose that the statement is true for all natural numbers $\leq p$. The binomial theorem yields the equations

$$(k+1)^{p+1} - k^{p+1} = (p+1)k^p + \text{terms involving lower powers of } k.$$

Adding for $k = 1,\ldots,n$, we obtain

$$\frac{(n+1)^{p+1}}{p+1} = \sum_{k=1}^{n} k^p + \text{terms involving } \sum_{k=1}^{n} k^r \text{ for } r < p.$$

By assumption, we can write each $\sum_{k=1}^{n} k^r$ as an expression involving powers $n^s$ with $s \leq p$. It follows that

$$\sum_{k=1}^{n} k^p = \frac{(n+1)^{p+1}}{p+1} + \text{terms involving powers of } n \text{ less than } p+1.$$

9. Suppose A contains 1, and that A contains $n + 1$ if it contains $n$. If A does not contain all natural numbers, then the set B of natural numbers not in A is not $\emptyset$. So B has a smallest member $n_0$. Now $n_0 \neq 1$, since A contains 1, so we can write $n_0 = (n_0 - 1) + 1$ where $n_0 - 1$ is a natural number. Now $n_0 - 1$ is not in B, so $n_0 - 1$ is in A. By hypothesis, $n_0$ must be in A, so $n_0$ is not in B, a contradiction. (By the way, the assertion that a natural number $n \neq 1$ can be written $n = m + 1$ for some other natural number $m$, can itself be proved by induction.)

10. Clearly 1 is in B. If $k$ is in B, then $1,\ldots,k$ are all in A, so $k + 1$ is in A, so $1,\ldots,k + 1$ are in A, so $k + 1$ is in B. By (ordinary) induction, $B = \mathbf{N}$, so also $A = \mathbf{N}$.

13. (a) If $\sqrt{2}+\sqrt{3}$ were rational, then $(\sqrt{2}+\sqrt{3})^2$ would certainly be rational. So $5+2\sqrt{6}$ would be rational, and consequently $\sqrt{6}$ would be rational, which is false.

(b) If $\sqrt{6}-\sqrt{2}-\sqrt{3}$ were rational, then the same would be true for

$$\begin{aligned}[\sqrt{6}-(\sqrt{2}+\sqrt{3})]^2 &= 6+(\sqrt{2}+\sqrt{3})^2-2\sqrt{6}(\sqrt{2}+\sqrt{3})\\ &= 11+2\sqrt{6}\,[1-(\sqrt{2}+\sqrt{3})].\end{aligned}$$

Thus $\sqrt{6}\,[1-(\sqrt{2}+\sqrt{3})]$ would be rational, so the same would be true for

$$\begin{aligned}\{\sqrt{6}\,[1-(\sqrt{2}+\sqrt{3})]\}^2 &= 6\,[1-(\sqrt{2}+\sqrt{3})]^2\\ &= 6\,[6-2(\sqrt{2}+\sqrt{3})+\sqrt{6}].\end{aligned}$$

So both

$$\sqrt{6}-(\sqrt{2}+\sqrt{3}) \quad \text{and} \quad \sqrt{6}-2(\sqrt{2}+\sqrt{3})$$

would be rational, which would imply that $\sqrt{2}+\sqrt{3}$ is rational, contradicting part (a).

14. (a) The assertion is true for $m=1$. If it is true for $m$, then

$$(p+\sqrt{q})^{m+1}=(p+\sqrt{q})(a+b\sqrt{q})=(ap+b)+(a+pb)\sqrt{q},$$

and $ap+b$ and $a+pb$ are rational.

(b) The assertion is true for $m=1$. If it is true for $m$, then

$$(p-\sqrt{q})^{m+1}=(p-\sqrt{q})(a-b\sqrt{q})=(ap+b)-(a+pb)\sqrt{q},$$

whereas $(p+\sqrt{q})^{m+1}=(ap+b)+(a+pb)\sqrt{q}$ by part (a).

15. (a) The inequality $(m + 2n)^2/(m + n)^2 > 2$ is equivalent to

$$m^2 + 4mn + 4n^2 > 2m^2 + 4mn + 2n^2,$$

or simply $2n^2 > m^2$.

The second inequality is equivalent to

$$n^2[(m + 2n)^2 - 2(m + n)^2] < (2n^2 - m^2)(m + n)^2,$$

or

$$n^2(2n^2 - m^2) < (2n^2 - m^2)(n^2 + [2mn + m^2]),$$

or

$$0 < (2n^2 - m^2)(2mn + m^2).$$

(b) Reverse all inequality signs in the solution for part (a).

(c) Let $m_1 = m + 2n$ and $n_1 = m + n$, and then choose

$$m' = m_1 + 2n_1 = 3m + 4n,$$

$$n' = m_1 + n_1 = 2m + 3n.$$

16. (a) Suppose that every number $< n$ can be written as a product of primes. If $n > 1$ is not a prime, then $n = ab$ for $a,b < n$. By assumption, a and b are each products of primes, so $n = ab$ is also.

(b) If $\sqrt{n} = a/b$, then $nb^2 = a^2$, so the factorization into primes of $nb^2$ and of $a^2$ must be the same. Now every prime appears an even number of times in the factorization of $a^2$, and of $b^2$, so the same must be true for the factorization of n. This implies that n is a square.

(c) Repeat the same argument, using the fact that every prime occurs a multiple of k times in $a^k$ and $b^k$.

(d) If $p_1,\dots,p_n$ were the only primes, then $p_1 \cdot p_2 \cdot \dots \cdot p_n + 1$ could not be a prime, since it is larger than all of them, (and is not 1), so it must be divisible by a prime. But $p_1,\dots,p_n$ clearly do not divide it, a contradiction. (Although this is a proof by contradiction, it can be used to obtain some positive information: If $p_1,\dots,p_n$ are the first n primes, then the $(n + 1)^{st}$ prime is $\leq p_1 \cdot p_2 \cdot \dots \cdot p_n + 1$. It is not necessarily true, however, that $p_1 \cdot p_2 \cdot \dots \cdot p_n + 1$ is a prime; for example, $2\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13 + 1 = 30{,}031 = 59\cdot 509$.)

17. (a) Suppose $x = p/q$ where p and q are natural numbers with no common factor. Then

$$\frac{p^n}{q^n} + a_{n-1}\frac{p^{n-1}}{q^{n-1}} + \dots + a_0 = 0,$$

so

$$(*) \quad p^n + a_{n-1}p^{n-1}q + \dots + a_0q^n = 0.$$

Now if $q \neq \pm 1$, then q has some prime number as a factor. This prime factor divides every term of (*) other than $p^n$, so it must divide $p^n$ also. Therefore it divides p, a contradiction. So $q = \pm 1$, which means that x is an integer.

(b) Writing the various powers of $x = 2^{2/6} + 2^{3/6}$ in terms of the powers of $\eta = 2^{1/6}$, we obtain the following table for the coefficients.

| | $\eta^0$ | $\eta$ | $\eta^2$ | $\eta^3$ | $\eta^4$ | $\eta^5$ |
|---|---|---|---|---|---|---|
| $x^0$ | 1 | | | | | |
| $x^1$ | | | 1 | 1 | | |
| $x^2$ | 2 | | | | 1 | 2 |
| $x^3$ | 2 | 6 | 6 | 2 | | |
| $x^4$ | | | 2 | 8 | 12 | 8 |
| $x^5$ | 40 | 40 | 20 | 4 | 2 | 10 |
| $x^6$ | 12 | 24 | 60 | 80 | 60 | 24 |

We can then find numbers $a_0,\ldots,a_5$ such that

$$x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0$$

by solving the equations $a_0 + 2a_2 + 2a_3 + 40a_5 + 12 = 0$, etc. It turns out that

$$x^6 - 6x^4 - 4x^3 + 12x^2 - 24x - 4 = 0.$$

Part (a) implies that either x is irrational or else x is an integer, and it is easy to see that x is not an integer, because $1.4 < \sqrt{2} < 1.5$ and $1.2 < \sqrt[3]{2} < 1.3$, so $2.6 < \sqrt{2} + \sqrt[3]{2} < 2.8$.

This is one of those problems where a little learning, though perhaps a dangerous thing, could save a lot of work: The proper equation for x can also be found by noting that $\sqrt{2} + \sqrt[3]{2}$ clearly satisfies the equation

$$[(x - \sqrt{2})^3 - 2]\cdot[(x + \sqrt{2})^3 - 2] = 0;$$

when the left side is multiplied out, we obtain

$$\begin{aligned}
&(x - 2)^3 + 4 - 2\cdot[(x - \sqrt{2})^3 + (x + \sqrt{2})^3] \\
&\quad = (x - 2)^3 + 4 - 2\cdot[2x^3 + 12x] \qquad \text{(the odd powers of x cancel out)} \\
&\quad = x^6 - 6x^4 - 4x^3 + 12x^2 - 24x - 4.
\end{aligned}$$

Of course, this method depends on the observation that the equation for $x = \sqrt{2} + \sqrt[3]{2}$ should also have $-\sqrt{2} + \sqrt[3]{2}$ as a root (a hint as to why this should be true will be found in Problem 24-8).

19. Since

$$\frac{\left(\frac{1 + \sqrt{5}}{2}\right)^1 - \left(\frac{1 - \sqrt{5}}{2}\right)^1}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1,$$

$$\frac{\left(\frac{1 + \sqrt{5}}{2}\right)^2 - \left(\frac{1 - \sqrt{5}}{2}\right)^2}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1,$$

the assertion is true for $n = 1$ and $n = 2$. Now suppose that the assertion is true for all $k < n$, where $n \geq 3$. Then it is true, in particular, for $n - 1$ and $n - 2$, so

$$\begin{aligned}
a_n &= a_{n-1} + a_{n-2} \\
&= \frac{\left(\frac{1 + \sqrt{5}}{2}\right)^{n-2} - \left(\frac{1 - \sqrt{5}}{2}\right)^{n-2} + \left(\frac{1 + \sqrt{5}}{2}\right)^{n-1} - \left(\frac{1 - \sqrt{5}}{2}\right)^{n-1}}{\sqrt{5}} \\
&= \frac{\left(\frac{1 + \sqrt{5}}{2}\right)^{n-2}\left(1 + \frac{1 + \sqrt{5}}{2}\right) - \left(\frac{1 - \sqrt{5}}{2}\right)^{n-2}\left(1 + \frac{1 - \sqrt{5}}{2}\right)}{\sqrt{5}} \\
&= \frac{\left(\frac{1 + \sqrt{5}}{2}\right)\left(\frac{1 + \sqrt{5}}{2}\right)^2 - \left(\frac{1 - \sqrt{5}}{2}\right)^{n-2}\left(\frac{1 - \sqrt{5}}{2}\right)^2}{\sqrt{5}} \\
&= \frac{\left(\frac{1 + \sqrt{5}}{2}\right)^n - \left(\frac{1 - \sqrt{5}}{2}\right)^n}{\sqrt{5}}.
\end{aligned}$$

20. (a) If $a_1 = \dots = a_n$, then equality holds. If $a_i$ and $a_j$ are replaced by $(a_i + a_j)/2$, the arithmetic mean $A_n$ is unchanged, while $G_n$ becomes

$$G_n' = \frac{G_n \cdot \sqrt[n]{\left(\frac{a_i + a_j}{2}\right)\left(\frac{a_i + a_j}{2}\right)}}{\sqrt[n]{a_i a_j}}$$

$$\geq G_n, \qquad \text{since } \left(\frac{a_i + a_j}{2}\right)^2 \geq a_i a_j \text{ by Problem 1-6.}$$

If this process is repeated enough times we finally have all $a_i$ equal, so there is a sequence of geometric means

$$G_n \leq G_n' \leq G_n'' \leq \cdots \leq G_n^{(k)} = A_n.$$

(b) We know that $G_n \leq A_n$ when $n = 2^1$. Suppose that $G_n \leq A_n$ for $n = 2^k$ and let $m = 2^{k+1} = 2n$. Then

$$\begin{aligned}
G_m &= \sqrt[m]{a_1 \cdot \ldots \cdot a_m} = \sqrt{\sqrt[n]{a_1 \cdot \ldots \cdot a_n}\ \sqrt[n]{a_{n+1} \cdot \ldots \cdot a_m}} \\
&\leq \frac{\sqrt[n]{a_1 \cdot \ldots \cdot a_n} + \sqrt[n]{a_{n+1} \cdot \ldots \cdot a_m}}{2} \qquad \text{using } G_2 \leq A_2 \\
&\leq \frac{\dfrac{a_1 + \cdots + a_n}{n} + \dfrac{a_{n+1} + \cdots + a_m}{n}}{2} \qquad \text{by assumption} \\
&= \frac{a_1 + \cdots + a_m}{2n} \\
&= A_m.
\end{aligned}$$

(c) Applying (b) to these $2^m$ numbers yields, for $k = 2^m - n$,

$$\begin{aligned}
(a_1 \cdot \ldots \cdot a_n)(A_n)^k &\leq \left[\frac{a_1 + \cdots + a_n + k\,A_n}{2^m}\right]^{2^m} \\
&= \left[\frac{nA_n + k\,A_n}{2^m}\right]^{2^m} = (A_n)^{2^m},
\end{aligned}$$

so

$$a_1\cdot\ldots\cdot a_n \le (A_n)^{2^m-k} = (A_n)^n.$$

21. Since $a^{n+1} = a^n\cdot a = a^n\cdot a^1$, the first equation is true for $m = 1$. Suppose that $a^{n+m} = a^n\cdot a^m$. Then

$$\begin{aligned} a^{n+(m+1)} &= a^{(n+m)+1} = a^{n+m}\cdot a && \text{by definition} \\ &= (a^n\cdot a^m)\cdot a \\ &= a^n\cdot(a^m\cdot a) \\ &= a^n\cdot a^{m+1} && \text{by definition,} \end{aligned}$$

so the first equation is true for $m + 1$.

Since $(a^n)^1 = a^n = a^{n\cdot 1}$, the second equation is true for $m = 1$. Suppose that $(a^n)^m = a^{nm}$. Then

$$\begin{aligned} (a^n)^{m+1} &= (a^n)^m\cdot a^n && \text{by definition} \\ &= a^{nm}\cdot a^n \\ &= a^{nm+n} && \text{by (i)} \\ &= a^{n(m+1)}. \end{aligned}$$

22. Since

$$\begin{aligned} 1\cdot(b + c) &= b + c && \text{by definition} \\ &= 1\cdot b + 1\cdot c && \text{by definition,} \end{aligned}$$

the first result is true for $a = 1$. Suppose that $a\cdot(b + c) = a\cdot b + a\cdot c$ for all $b$ and $c$. Then

$$\begin{aligned}(a + 1)\cdot(b + c) &= a\cdot(b + c) + (b + c) && \text{by definition}\\ &= (a\cdot b + a\cdot c) + (b + c)\\ &= (a\cdot b + b) + (a\cdot c + c) && \text{by P1 and P4}\\ &= (a + 1)\cdot b + (a + 1)\cdot c && \text{by definition.}\end{aligned}$$

The equation $a\cdot 1 = a$ is true for $a = 1$ by definition. Suppose that $a\cdot 1 = a$. Then

$$\begin{aligned}(a + 1)\cdot 1 &= a\cdot 1 + 1\cdot 1 && \text{by definition}\\ &= a + 1.\end{aligned}$$

For $b = 1$, the equation $a\cdot b = b\cdot a$ follows from $a\cdot 1 = a$, which has just been proved, and $1\cdot a = a$, which is true by definition. Suppose that $a\cdot b = b\cdot a$. Then

$$\begin{aligned}a\cdot(b + 1) &= a\cdot b + a\cdot 1\\ &= a\cdot b + a\\ &= b\cdot a + a\\ &= (b + 1)\cdot a && \text{by definition.}\end{aligned}$$

23. (a) (i) is clear.

(ii) This is clear, because 1 is positive, and if $k$ is positive, then $k + 1$ is positive.

(iii) Clearly 1 is in this set. If condition (2) failed for this set, then there would be some $k$ in the set with $k + 1 = 1/2$. But this is false, since $k = -1/2$ is not positive.

(iv) This set contains 4 but not 4 + 1.

(v) Since 1 is in A and B, also 1 is in C. If k is in C, then k is in both A and B, so k + 1 is in A and B, so k + 1 is in C.

(b) (i) 1 is a natural number because 1 is in every inductive set, by definition of inductive sets.

(ii) If k is a natural number, then k is in every inductive set. So k + 1 is in every inductive set. So k + 1 is a natural number.

# CHAPTER 3

1. (ii) $x/(x + 1)$ (for $x \neq 0, -1$).

   (iv) $1/(1 + x + y)$ (for $x + y \neq -1$).

   (vi) For all $c$, since $f(c \cdot 0) = f(0)$.

2. (ii) Rational $y$ between $-1$ and $1$, and all $y$ with $|y| > 1$.

   (iv) All $w$ with $0 \leq w \leq 1$.

3. (ii) $\{x: -1 \leq x \leq 1\}$.

   (iv) $\{-1, 1\}$.

4. (ii) $\sin^2 y$.

   (iv) $\sin t^3$.

5. (ii) $s \circ P$.

   (iv) $S \circ s$.

   (vi) $s \circ (P + P \circ S)$.

   (viii) $P \circ S \circ s + s \circ S + P \circ s \circ (S + s)$.

6. (a) Let

$$f_i(x) = \frac{\prod\limits_{\substack{j=1\\ j\neq i}}^{n} (x - x_j)}{\prod\limits_{\substack{j=1\\ j\neq i}}^{n} (x_i - x_j)}.$$

(b) Let

$$f(x) = \sum_{j=1}^{n} a_i f_i(x)$$

$$= \sum_{i=1}^{n} a_i \cdot \prod_{\substack{j=1\\ j\neq 1}}^{n} \frac{(x - x_j)}{(x_i - x_j)}.$$

7. (a) If the degree of f is 1, then f is of the form

$$f(x) = cx + d = c(x - a) + (d + ac),$$

so we can let $g(x) = c$ and $b = d + ac$. Suppose that the result is true for polynomials of degree $\leq k$. If f has degree $k + 1$, then f has the form

$$f(x) = a_{k+1}x^{k+1} + \ldots + a_1x + a_0.$$

Now the polynomial function $h(x) = f(x) - a_{k+1}(x - a)$ has degree $\leq k$, so we can write

$$f(x) - a_{k+1}(x - a) = (x - a)g(x) + b,$$

or

$$f(x) = (x - a)[g(x) + a_{k+1}] + b,$$

which is the required form.

(b) By part (a), we can write $f(x) = (x - a)g(x) + b$. Then

$$0 = f(a) = (a - a)g(a) + b = b,$$

so $f(x) = (x - a)g(x)$.

(c) Suppose f has n roots $a_1,\dots,a_n$. Then by part (b) we can write $f(x) = (x - a)g_1(x)$ where the degree of $g_1(x)$ is $n - 1$. Now

$$0 = f(a_2) = (a_2 - a_1)g_1(a_2),$$

so $g_1(a_2) = 0$, since $a_2 \neq a_1$. Thus we can write

$$f(x) = (x - a_1)(x - a_2)g_2(x),$$

where the degree of $g_2$ is $n - 2$. Continuing in this way, we find that

$$f(x) = (x - a_1)(x - a_2)\cdot\ldots\cdot(x - a_n)c$$

for some number $c \neq 0$. It is clear that $f(a) \neq 0$ if $a \neq a_1,\dots,a_n$. So f can have at most n roots.

(d) If $f(x) = (x - 1)(x - 2)\cdot\ldots\cdot(x - n)$, then f has n roots. If n is even, then $f(x) = x^n + 1$ has no roots. If n is odd, then $f(x) = x^n$ has only one root, namely 0.

8. If

$$x = f(f(x)) = \frac{a\left(\frac{ax + b}{cx + d}\right) + b}{c\left(\frac{ax + b}{cx + d}\right) + d}$$

for all x, then

$$(ac + cd)x^2 + (d^2 - a^2)x - ab - bd = 0 \qquad \text{for all } x,$$

so

$$ac + cd = 0,$$

$$ab + bd = 0,$$

$$d^2 - a^2 = 0.$$

It follows that $a = d$ or $a = -d$. One possibility is $a = d = 0$, in which case $f(x) = b/(cx)$, which satisfies $f(f(x)) = x$ for all $x \neq 0$. If $a = d \neq 0$, then $b = c = 0$, so $f(x) = x$. The third possibility is $a + d = 0$, so that $f(x) = (ax + b)/(cx - a)$, which satisfies $f(f(x)) = x$ for all $x \neq a/c$ (strictly speaking we should add the proviso that $f(x) \neq a/c$ for $x \neq a/c$, which means that

$$\frac{ax + b}{cx - a} \neq \frac{a}{c},$$

or $a^2 + bc \neq 0$).

9. (a)

$$C_{A \cap B} = C_A \cdot C_B,$$

$$C_{\mathbf{R} - A} = 1 - C_A,$$

$$C_{A \cup B} = C_A + C_B - C_A \cdot C_B.$$

(b) Let $A = \{x: f(x) = 1\}$.

(c) $f = f^2$ if and only if $f(x) = 0$ or $1$ for all $x$; so part (b) may be applied.

10. (a) Those functions $f$ satisfying $f(x) \geq 0$ for all $x$.

(b) Those functions $f$ with $f(x) \neq 0$ for all $x$.

(c) Those functions $b$ and $c$ satisfying $(b(t))^2 - 4c(t) \geq 0$ for all $t$.

(d) $b(t)$ must $= 0$ whenever $a(t) = 0$. If $a(t) \neq 0$ for all $t$, then there is a unique such function, namely $x(t) = a(t)/b(t)$. If $a(t) = 0$ for some $t$, then $x(t)$ can be chosen arbitrarily, so there are infinitely many such $x$.

11. (d) Let $H(1)$, $H(2)$, $H(13)$, $H(36)$, $H(\pi/3)$, and $H(47)$ have the values already prescribed, and let $H(x) = 0$ for $x \neq 1, 2, 13, 36, \pi/3, 47$. Since, in particular, $H(0) = 0$, the equation $H(H(x)) = H(x)$ holds for all $x$.

(e) Let $H(1) = 7$, $H(7) = 7$, $H(17) = 18$, $H(18) = 18$, and $H(x) = 0$ for $x \neq 1, 7, 17, 18$.

13. (a) Let

$$E(x) = \frac{f(x) + f(-x)}{2}, \qquad O(x) = \frac{f(x) - f(-x)}{2}.$$

(b) If $f = E + O$, when $E$ is even and $O$ is odd, then

$$f(x) = E(x) + O(x),$$
$$f(-x) = E(x) - O(x).$$

Solving, we obtain the above expressions for $E(x)$ and $O(x)$.

14. $\max(f,g) = (f + g + |f - g|)/2$; $\min(f,g) = (f + g - |f - g|)/2$. (See Problem 1-13.)

15. (a) $f = \max(f,0) + \min(f,0)$ because $f(x) = \max(f(x),0) + \min(f(x),0)$ for all $x$, the equation $a = \max(a,0) + \min(a,0)$ holding for all numbers $a$.

(b) For each $x$, choose numbers $g(x), h(x) \geq 0$ with $f(x) = g(x) - h(x)$. Since we can choose each pair $g(x)$ and $h(x)$ in infinitely many ways, there are infinitely many such functions $g$ and $h$.

16. (a) The result is true for $n = 1$. If $f(x_1 + \dots + x_n) = f(x_1) + \dots + f(x_n)$ for all $x_1,\dots,x_n$, then

$$\begin{aligned} f(x_1 + \dots + x_{n+1}) &= f([x_1 + \dots + x_n] + x_{n+1}) \\ &= f(x_1 + \dots + x_n) + f(x_{n+1}) \\ &= f(x_1) + \dots + f(x_n) + f(x_{n+1}). \end{aligned}$$

(c) Let $c = f(1)$. Now for any natural number $n$,

$$f(n) = f(\underbrace{1 + \dots + 1}_{n \text{ times}}) = \underbrace{f(1) + \dots + f(1)}_{n \text{ times}} = cn.$$

Since

$$f(x) + f(0) = f(x + 0) = f(x),$$

it follows that $f(0) = 0$. Then since

$$f(x) + f(-x) = f(x + (-x)) = f(0) = 0,$$

it follows that $f(-x) = -f(x)$. In particular, for any natural number $n$,

$$f(-n) = -f(n) = -cn = c(-n).$$

Moreover,

$$\underbrace{f\left(\frac{1}{n}\right) + \dots + f\left(\frac{1}{n}\right)}_{n \text{ times}} = f\left(\underbrace{\frac{1}{n} + \dots + \frac{1}{n}}_{n \text{ times}}\right) = f(1) = c,$$

so

$$f\left(\frac{1}{n}\right) = c\cdot\frac{1}{n},$$

and consequently

$$f\left(\frac{1}{-n}\right) = f\left(-\frac{1}{n}\right) = -f\left(\frac{1}{n}\right) = -c\cdot\frac{1}{n} = c\left(\frac{1}{-n}\right).$$

Finally, any rational number can be written m/n for m a natural number, and n an integer; and

$$f\left(\frac{m}{n}\right) = f\underbrace{\left(\frac{1}{n} + \dots + \frac{1}{n}\right)}_{\text{m times}} = \underbrace{f\left(\frac{1}{n}\right) + \dots + f\left(\frac{1}{n}\right)}_{\text{m times}}$$

$$= mc\cdot\frac{1}{n} = c\cdot\frac{m}{n}.$$

17. (a) Since $f(a) = f(a\cdot 1) = f(a)\cdot f(1)$ and $f(a) \neq 0$ for some a, we have $f(1) = 1$.

(b) According to Problem 16, $f(x) = f(1)x = x$ for all rational x.

(c) If $c > 0$, then $c = d^2$ for some d, so $f(c) = f(d^2) = (f(d))^2 \geq 0$. Moreover, we cannot have $f(c) = 0$, since this would imply that

$$f(a) = f\left(c\cdot\frac{a}{c}\right) = f(c)\cdot f\left(\frac{a}{c}\right) = 0 \qquad \text{for all a.}$$

(d) If $x > y$, then $x - y > 0$, so $f(x) - f(y) > 0$, by part (c).

(e) Suppose that $f(x) > x$ for some x. Choose a rational number r with $x < r < f(x)$. Then, by parts (b) and (d),

$$f(x) < f(r) = r < f(x),$$

a contradiction. Similarly, it is impossible that $f(x) < x$. (There is a minor detail here which requires justification. See Problem 8-5.)

8. If either $f = 0$ or $g = 0$, and also either $h = 0$ or $k = 0$, then the equation certainly holds. If not, then there is some $x$ with $f(x) \neq 0$, and some $y$ with $g(y) \neq 0$. Then $0 \neq f(x)g(y) = h(x)k(y)$, so we also have $h(x) \neq 0$ and $k(y) \neq 0$. Letting $\alpha = h(x)/f(x)$, we have $g(y') = \alpha k(y')$ for all $y'$. Moreover $\alpha = g(y)/k(y)$, so we also have $h(x') = \alpha f(x')$ for all $x'$. Thus we have $g = \alpha k$ and $h = \alpha f$ for some number $\alpha \neq 0$.

. (a) (i) If $f(x) + g(y) = xy$ for all $x$ and $y$, then, in particular,

$$f(x) + g(0) = 0 \qquad \text{for all } x.$$

So $f(x) = -g(0)$ for all $x$, and

$$-g(0) + g(y) = xy \qquad \text{for all } y;$$

setting $x = 0$ we obtain $g(y) = g(0)$. So we must have

$$0 = -g(0) + g(0) = xy \qquad \text{for all } x \text{ and } y,$$

which is absurd.

(ii) Setting $x = 0$, we obtain $f(y) = y$ for all $y$. So

$$x + y = g(x) - y \qquad \text{for all } y.$$

So $g(x) = x$ for all $x$. Thus

$$x + y = x - y \qquad \text{for all } x \text{ and } y,$$

which is absurd.

(b) Let f and g be the same constant function. (Arguments similar to those used in part (a) show that these are the only possible choices.)

20. (a) Let $f(x) = x$.

(b) For every natural number n we have

$$\begin{aligned} |f(y) - f(x)| &= \left|\sum_{k=1}^{n} f(x + \tfrac{k}{n}[y - x]) - f(x + \tfrac{k-1}{n}[y - x])\right| \\ &\leq \sum_{k=1}^{n} \left|f(x + \tfrac{k}{n}[y - x]) - f(x + \tfrac{k-1}{n}[y - x])\right| \\ &\leq \sum_{k=1}^{n} \frac{1}{n^2}(y - x)^2 \\ &= \frac{(y - x)^2}{n}. \end{aligned}$$

Therefore $f(y) = f(x)$ for all x and y.

22. (a) If $f(x) = f(y)$, then $g(x) = h(f(x)) = h(f(y)) = g(y)$.

(b) If $z = f(x)$, define $h(z) = g(x)$. This definition makes sense, because if $z = f(x')$, then $g(x) = g(x')$ by part (a). For z not of the form $f(x)$, define h any old way (or leave it undefined). Then for all x in the domain of f we have $g(x) = h(f(x))$.

23. (a) Suppose $x \neq y$. Then $g(x) = g(y)$ would imply that $x = f(g(x)) = f(g(y)) = y$, a contradiction.

(b) $b = f(g(b))$, so let $a = g(b)$.

24. (a) The hypothesis can be stated as follows: If $x = y$, then $g(x) = g(y)$. The conclusion now follows from Problem 22(b), applied to g and I.

(b) For each x, choose some number a such that $x = f(a)$. Call this number $g(x)$. Then $f(g(x)) = x = I(x)$ for all x.

25. It suffices to find a function f such that $f(x) \neq f(y)$ if $x \neq y$, but such that not every number is of the form $f(x)$, because by Problem 24(a) there will be a function g with $g \circ f = I$, and by Problem 23(b) there will not be a function g with $f \circ g = I$. One such function is

$$f(x) = \begin{cases} x, & x \leq 0 \\ x + 1, & x > 0; \end{cases}$$

no number between 0 and 1 is of the form $f(x)$.

26. $h \circ f \circ g = h \circ (f \circ g) = h \circ I = h$, and also $h \circ f \circ g = (h \circ f) \circ g = I \circ g = g$.

27. (a) The condition $f \circ g = g \circ f$ means that $g(x) + 1 = g(x + 1)$ for all x. There are many such g. In fact, g can be defined arbitrarily for $0 \leq x < 1$, and its values for other x determined from this equation.

(b) If $f(x) = c$ for all $x$, then $f \circ g = g \circ f$ if and only if $c = f(g(x)) = g(f(x)) = g(c)$, i.e., $c = g(c)$.

(c) If $f \circ g = g \circ f$ for all $g$, then in particular this is true for all constant functions $g(x) = c$. It follows from part (b) that $f(c) = c$ for all $c$.

28. (a) is a straightforward check.

(b) Let $f$ be a function with $f(x) = 0$ for some $x$, but not all $x$. Then $f \neq 0$, but there is clearly no function $g$ with $f(x) \cdot g(x) = 1$ for all $x$.

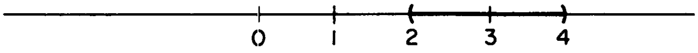
(c) Let $f$ and $g$ be the two functions which are 0 except at $x_0$ and $x_1$, with $f(x_0) = 1$, $f(x_1) = 0$ and $g(x_0) = 0$, $g(x_1) = 1$. Neither is 0, so $f$ or $-f$ would have to be in P, and likewise $g$ or $-g$. But $(\pm f)(\pm g) = 0$, which contradicts P12.

(d) P'11, P'12, and P'13 are true. P'10 is false; although at most one of the conditions holds, it is not necessarily true that at least one holds. For example, if $f(x) > 0$ for some $x$ and $< 0$ for other $x$, then neither $f = 0$, $f < 0$, nor $0 < f$ is true.

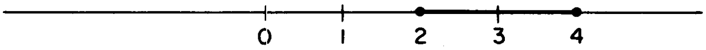
(e) No for the first example; if $h(x) = -x$, then $f < g$ actually implies that $h \circ f > h \circ g$. Yes for the second; for $f(h(x)) < g(h(x))$ for all $x$.

# CHAPTER 4

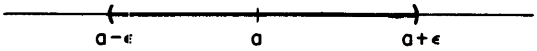
1. (i) $(2, 4)$.



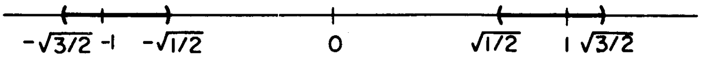
(ii) $[2, 4]$.



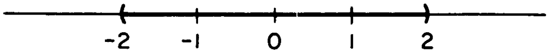
(iii) $(a - \epsilon, a + \epsilon)$.



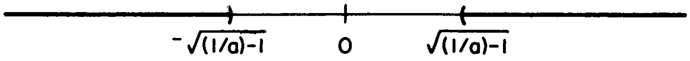
(iv) $(-\sqrt{3/2}, -\sqrt{1/2}) \cup (\sqrt{1/2}, \sqrt{3/2})$.



(v) $(-2, 2)$.



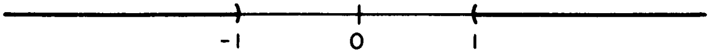
(vi) $\emptyset$ if $a \leq 0$;



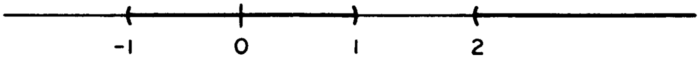
$\mathbf{R}$ if $a \geq 1$;

$(-\infty, -\sqrt{(1/a) - 1}] \cup [\sqrt{(1/a) - 1}, \infty)$ if $0 < a < 1$.

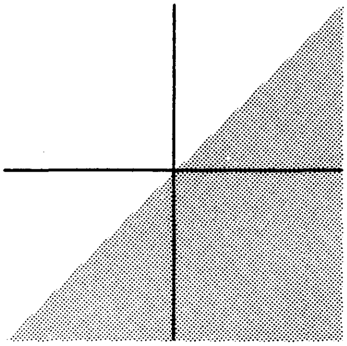
(vii) $(-\infty, 1] \cup [1,\infty)$.



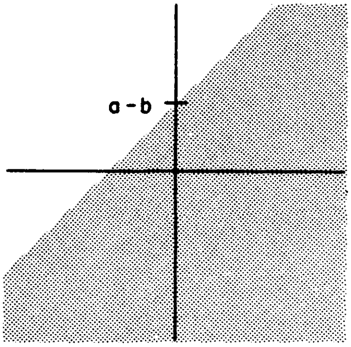
(viii) $(-1, 1) \cup (2, \infty)$.



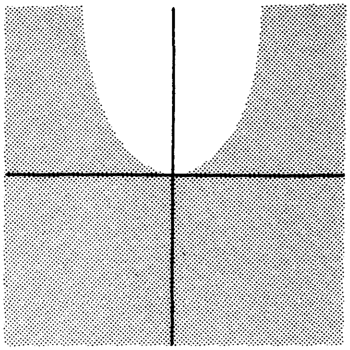
2. (i)



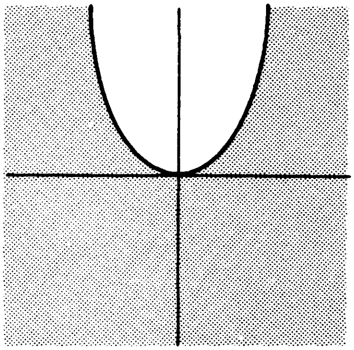
(ii)



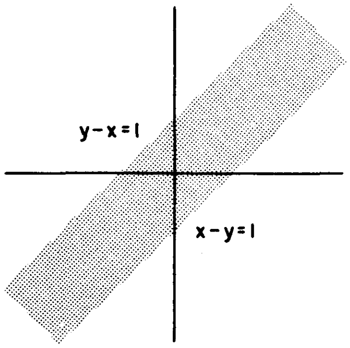
(iii)



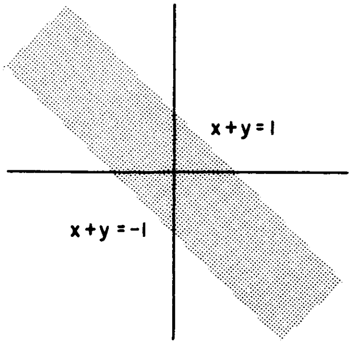
(iv)



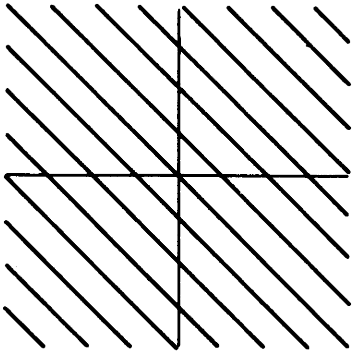
(v)



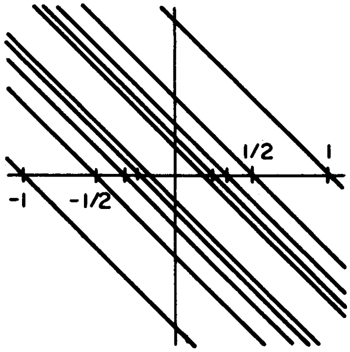
(vi)



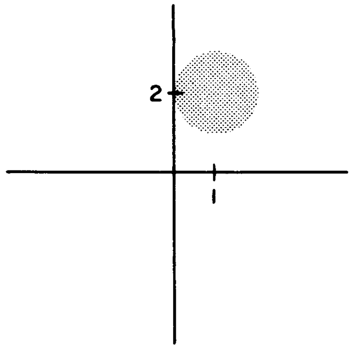
(vii)



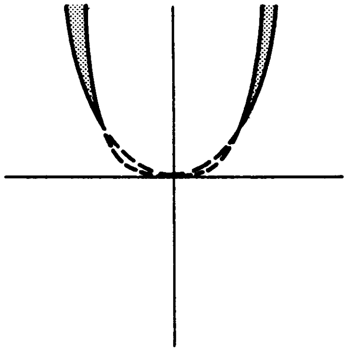
(viii)



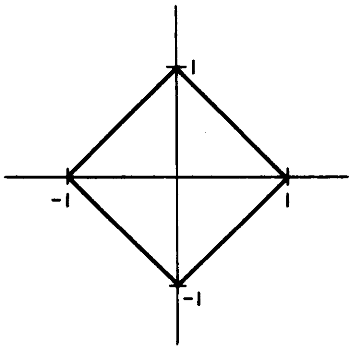
(ix)



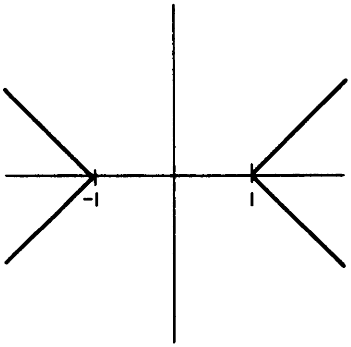
(x)



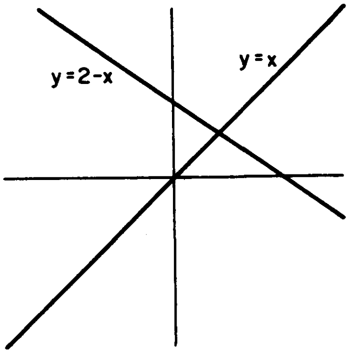
3. (i)



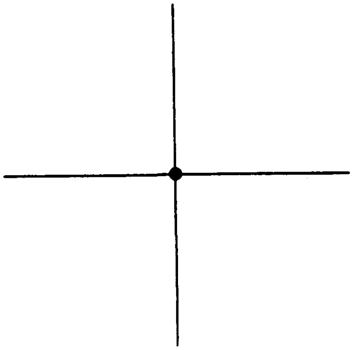
(ii)



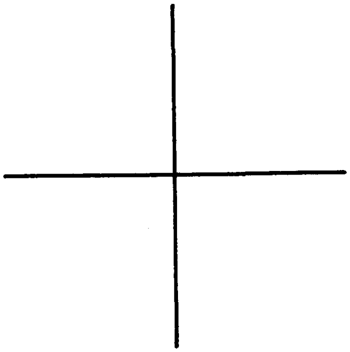
(iii), (iv)



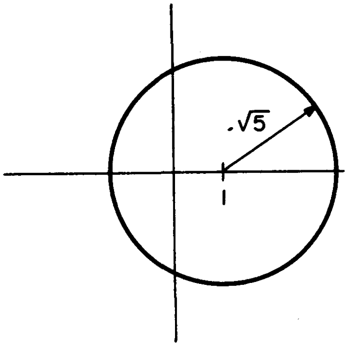
(v)



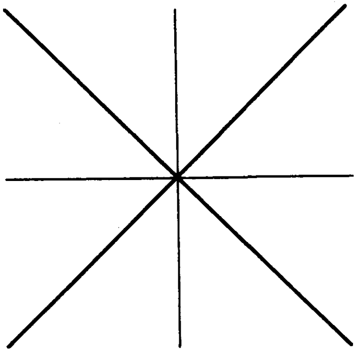
(vi)



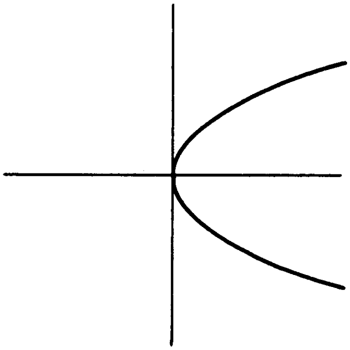
(vii) $x^2 - 2x + y^2 = (x - 1)^2 + y^2 - 1$



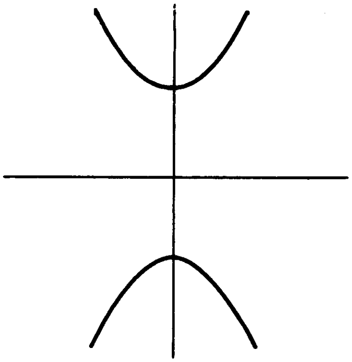
(viii)



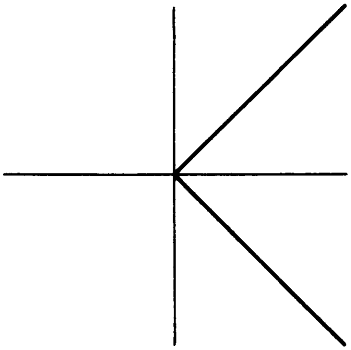
4. (i)



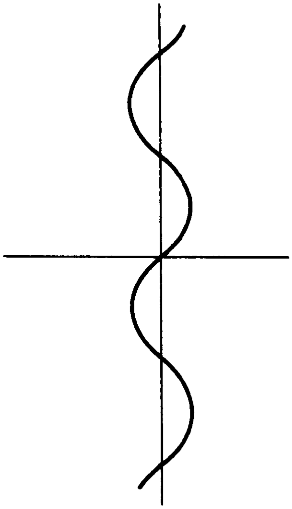
(ii)



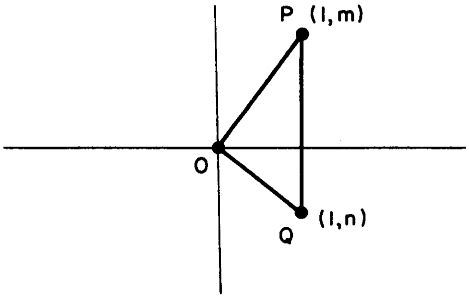
(iii)



(iv)



7. (a) The angle POQ is a right angle if and only if $(PQ)^2 = (PO)^2 + (OQ)^2$.



This means that

$$(m - n)^2 = m^2 + 1 + n^2 + 1,$$

which is equivalent to $-2mn = 2$, or $mn = -1$. This proves the result when $b = c = 0$. The general case follows from this special case, since perpendicularity depends only on the slope.

(b) If $B \neq 0$ and $B' \neq 0$, these straight lines are the graphs of

$$f(x) = (-A/B)x - C/A,$$
$$g(x) = (-A'/B')x - C/A;$$

so, by part (a), the lines are perpendicular if and only if

$$\left(-\frac{A}{B}\right)\cdot\left(-\frac{A'}{B'}\right) = -1,$$

which is equivalent to $AA' + BB' = 0$. If $B = 0$ (and consequently $A \neq 0$), then the first line is vertical, so the second is perpendicular to it if and only if $A' = 0$, which happens precisely when $AA' + BB' = 0$. Similarly if $B' = 0$.

8. (a) This inequality is equivalent to the squared inequality,

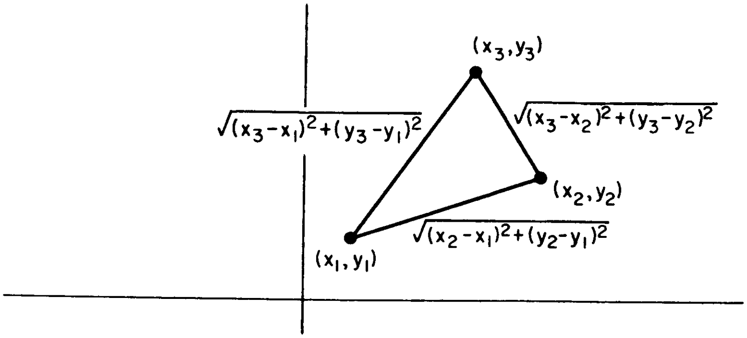
$$(x_1 + y_1)^2 + (x_2 + y_2)^2 \leq (x_1{}^2 + x_2{}^2) + (y_1{}^2 + y_2{}^2) + 2\sqrt{x_1{}^2 + x_2{}^2}\,\sqrt{y_1{}^2 + y_2{}^2},$$

which is easily seen to be equivalent to the Schwartz inequality.

(b) In part (a), replace

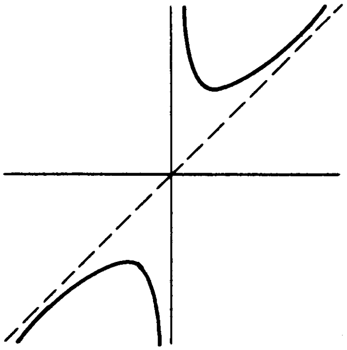
$$\begin{aligned} x_1 &\text{ by } x_2 - x_1, \\ x_2 &\text{ by } y_2 - y_1, \\ y_1 &\text{ by } x_3 - x_2, \\ y_2 &\text{ by } y_3 - y_2. \end{aligned}$$

Geometrically, this inequality says that the length of one side of a triangle is less than the sum of the lengths of the other two. (Notice that the additional information about the Schwartz inequality which was presented in Problem 1-18(d) shows that $\leq$ can be replaced by $<$ in the triangle inequality except when $(x_1, y_1)$, $(x_2, y_2)$, and $(x_3, y_3)$ lie on a straight line.)

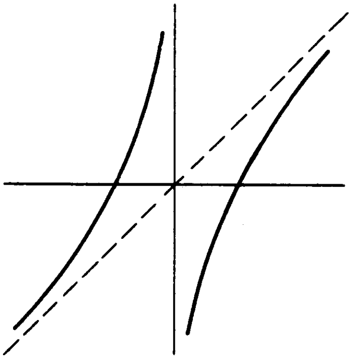

$(x_3,y_3)$
$\sqrt{(x_3-x_1)^2+(y_3-y_1)^2}$
$\sqrt{(x_3-x_2)^2+(y_3-y_2)^2}$
$(x_2,y_2)$
$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$
$(x_1,y_1)$

9. (The following figures do not indicate any particular points, since they were drawn using the method of Chapter 11, rather than by plotting points.)

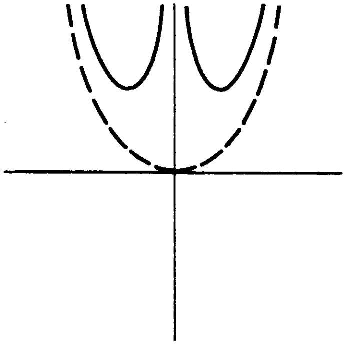
(i) This function is odd.



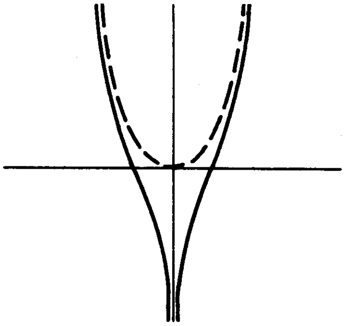
(ii) This function is odd.



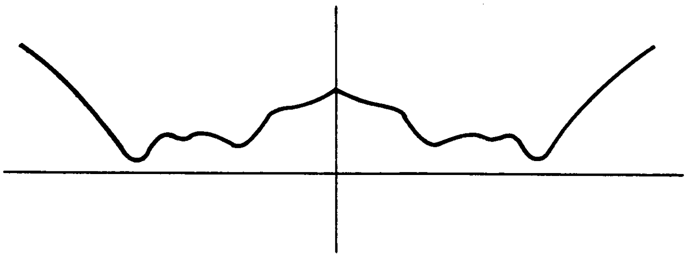
(iii) This function is even.



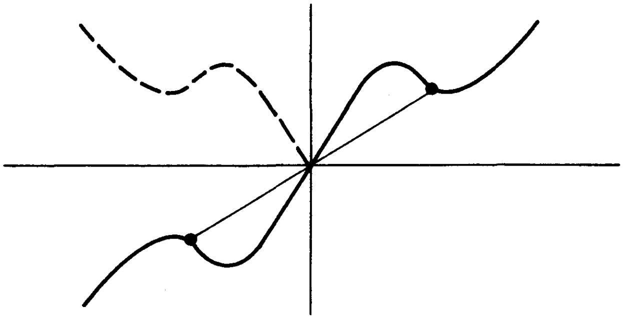
(iv) This function is even.



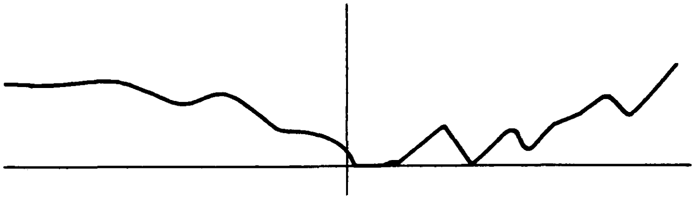
10. (i) The graph of f is symmetric with respect to the vertical axis.



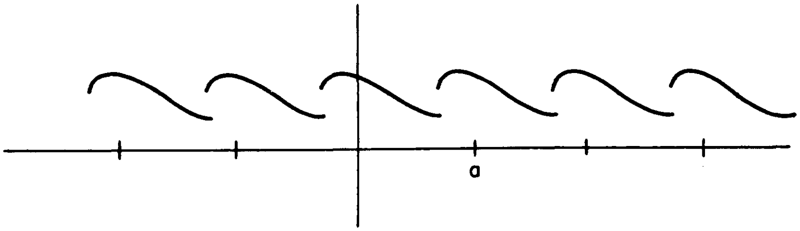
(ii) The graph of f is symmetric with respect to the origin. Equivalently, the part of the graph to the left of the vertical axis is obtained by reflecting first through the vertical axis, and then through the horizontal axis.



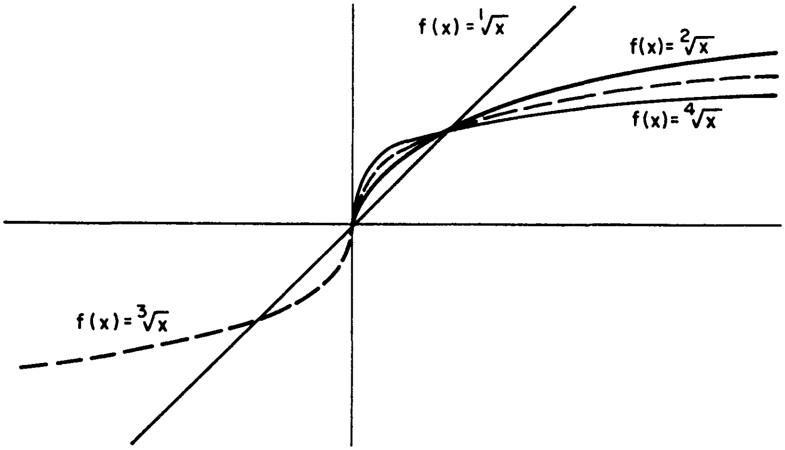
(iii) The graph of f lies above or on the horizontal axis.



(iv) The graph of f repeats the part between 0 and a over and over.

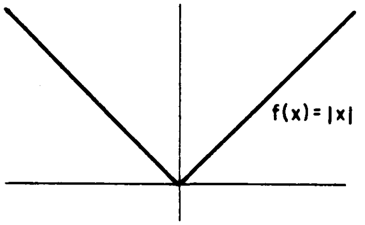


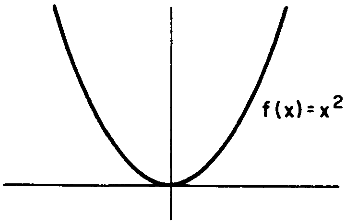
11. When n is odd, the domain of f is **R** , but when n is even, the domain of f is [0, ∞).



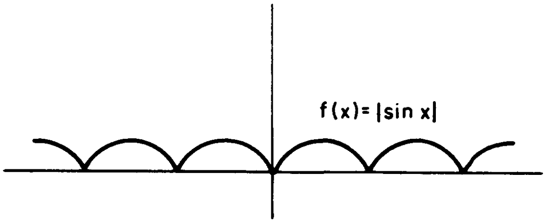
12. The graphs of $f(x) = |x|$ and $f(x) = |\sin x|$ contain "corners".

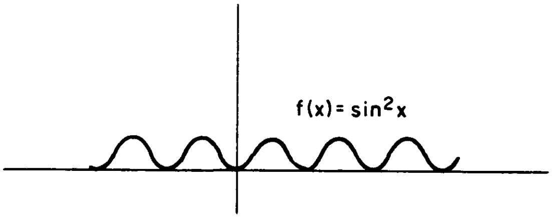
(a)



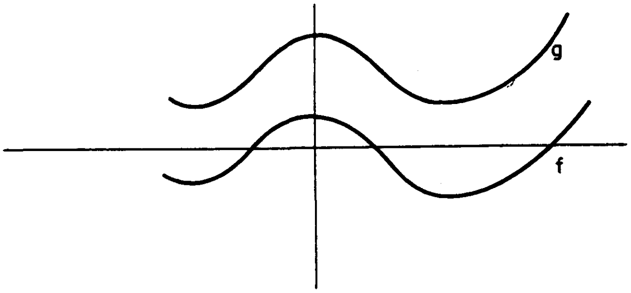


(b)

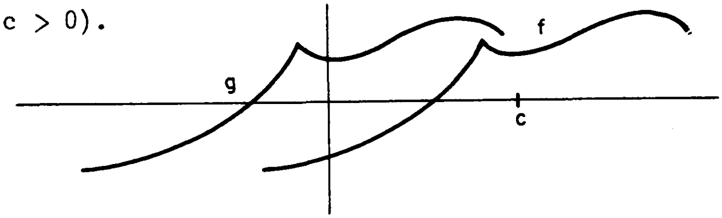




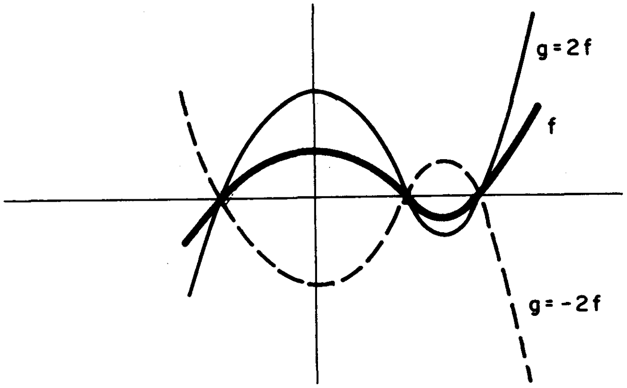
13. (i) The graph of $g$ is the graph of $f$ moved up $c$ units.



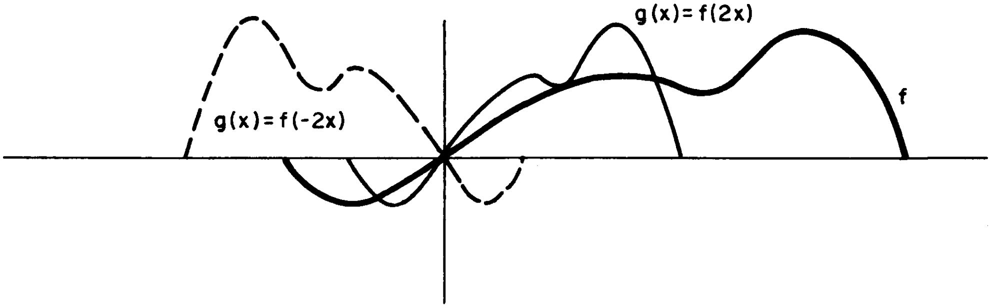
(ii) The graph of $g$ is the graph of $f$ moved over $c$ units to the left (if $c > 0$).



(iii) The height of the graph of $f$ is multiplied by a factor of $c$ everywhere. If $c = 0$, this means that $g = 0$; if $c > 0$, distances from the horizontal are increased in the same direction; if $c < 0$, distances are increased, but directions are changed.

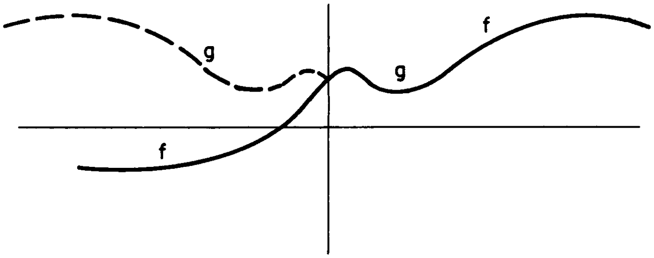


(iv) The graph of f is compressed by a factor of c if c > 0; if c < 0, the compression is combined with reflection through the vertical axis. If c = 0, then g is a constant function, $g(x) = f(0)$.

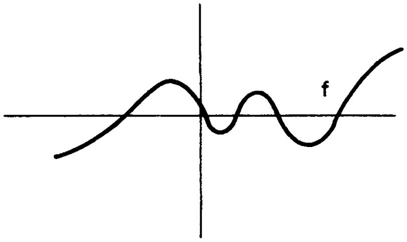


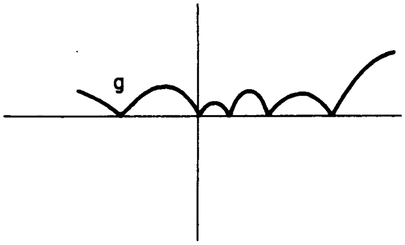
(v) "Everything that happens far out happens near 0, and vice versa," amply illustrated by the graph of $g(x) = \sin(1/x)$.

(vi) The graph of g consists of the part of the graph to the right of the vertical axis, together with its reflection through the vertical axis.

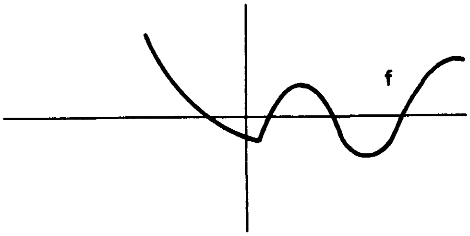


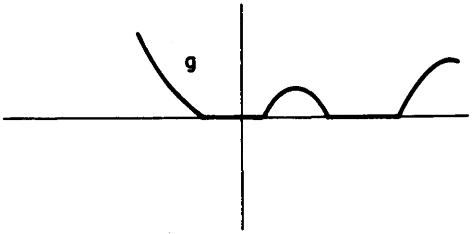
(vii) The graph of g is obtained by flipping up any parts of the graph of f which lie below the horizontal axis.



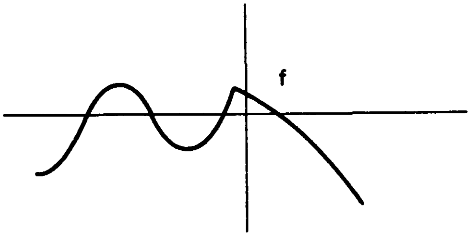


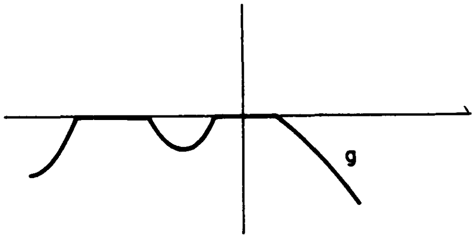
(viii) The graph of g is obtained by "cutting off" the parts of the graph of f which lies below the horizontal axis.



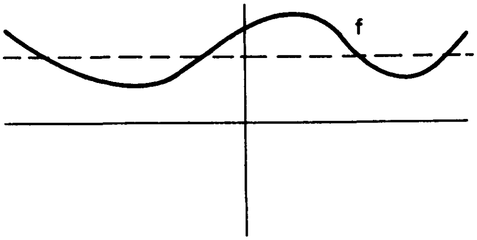


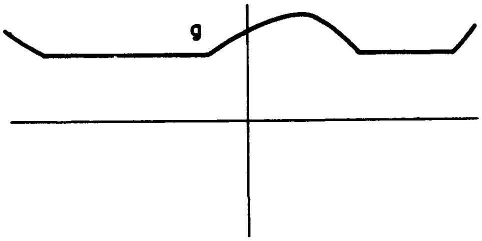
(ix) The graph of g is obtained by "cutting cff" the part of the graph of f which lies above the horizontal axis.





(x) The graph of g is obtained by "cutting off" the part of the graph of f which lies below the horizontal line at height 1.

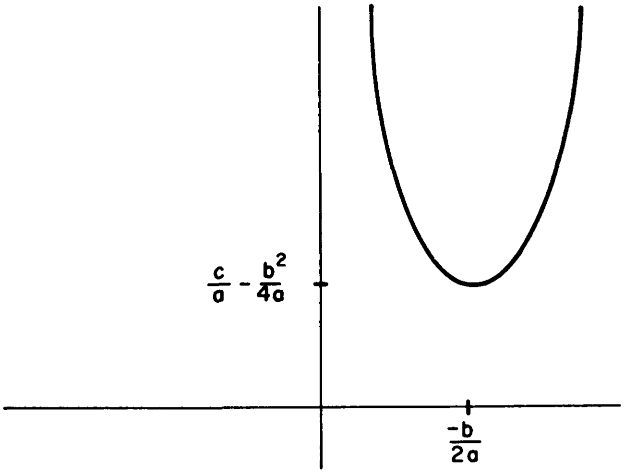




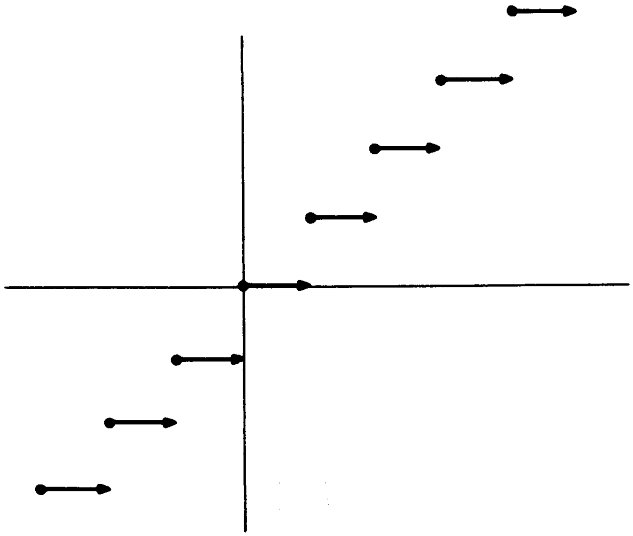
14. Since

$$f(x) = ax^2 + bx + c = a(x^2 + \frac{b}{a}x + \frac{c}{a})$$
$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \left(\frac{c}{a} - \frac{b^2}{4a}\right)\right],$$

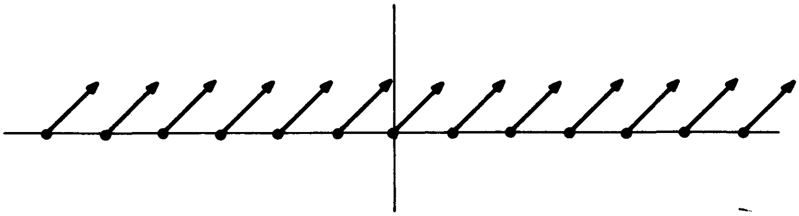
the graph looks like the figure below.



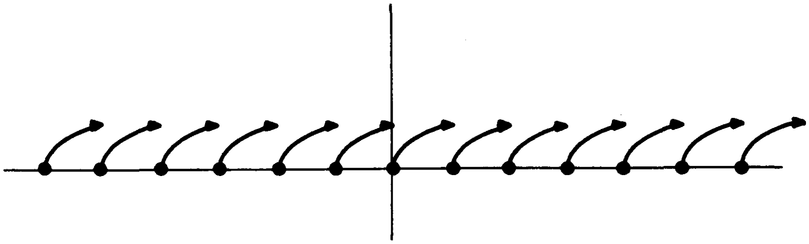
15. (i)



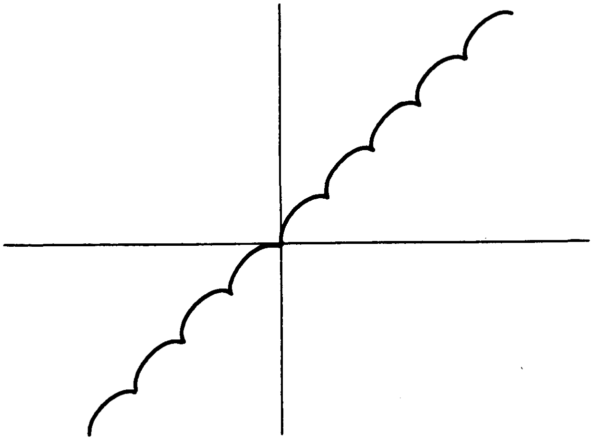
(ii)



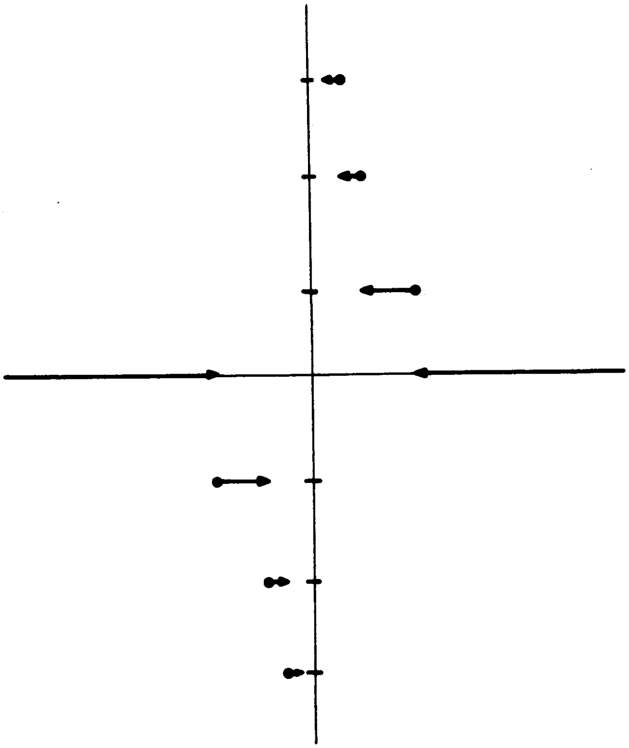
(iii)



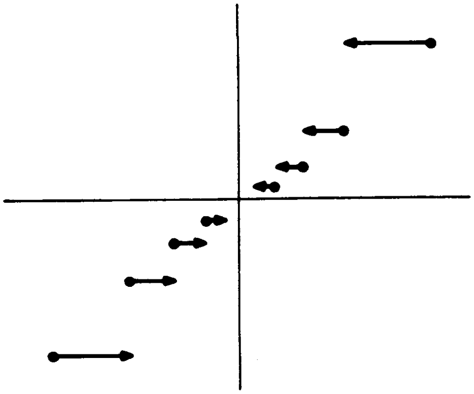
(iv)



(v)

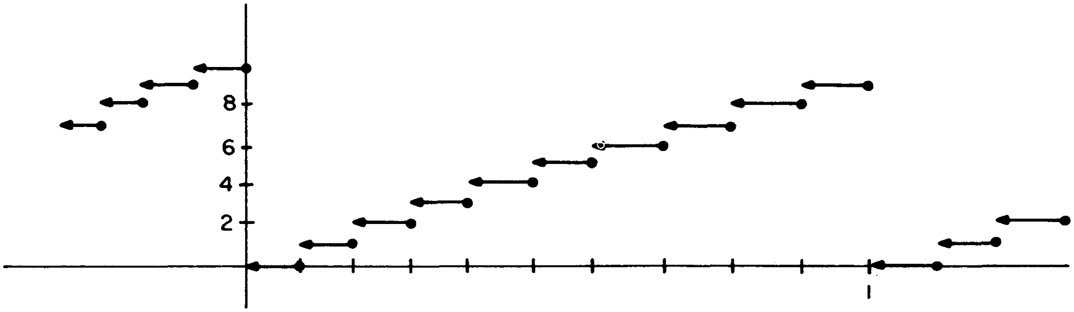


(vi) Notice that the domain of f is $\{x: -1 \leq x \leq 1 \text{ and } x \neq 0\}$.



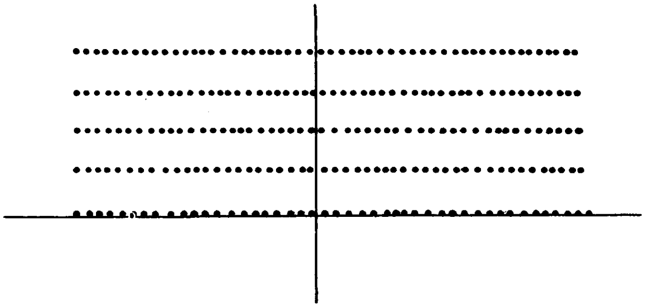
16. See pages 421 and 423 of the text.

17. (i) Notice that different scales have been used on the two axes.

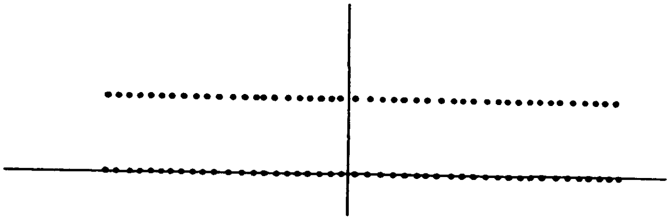


(ii) The graph of f is similar to the graph in part (i), except that there are ten sets of ten steps between n and n + 1.

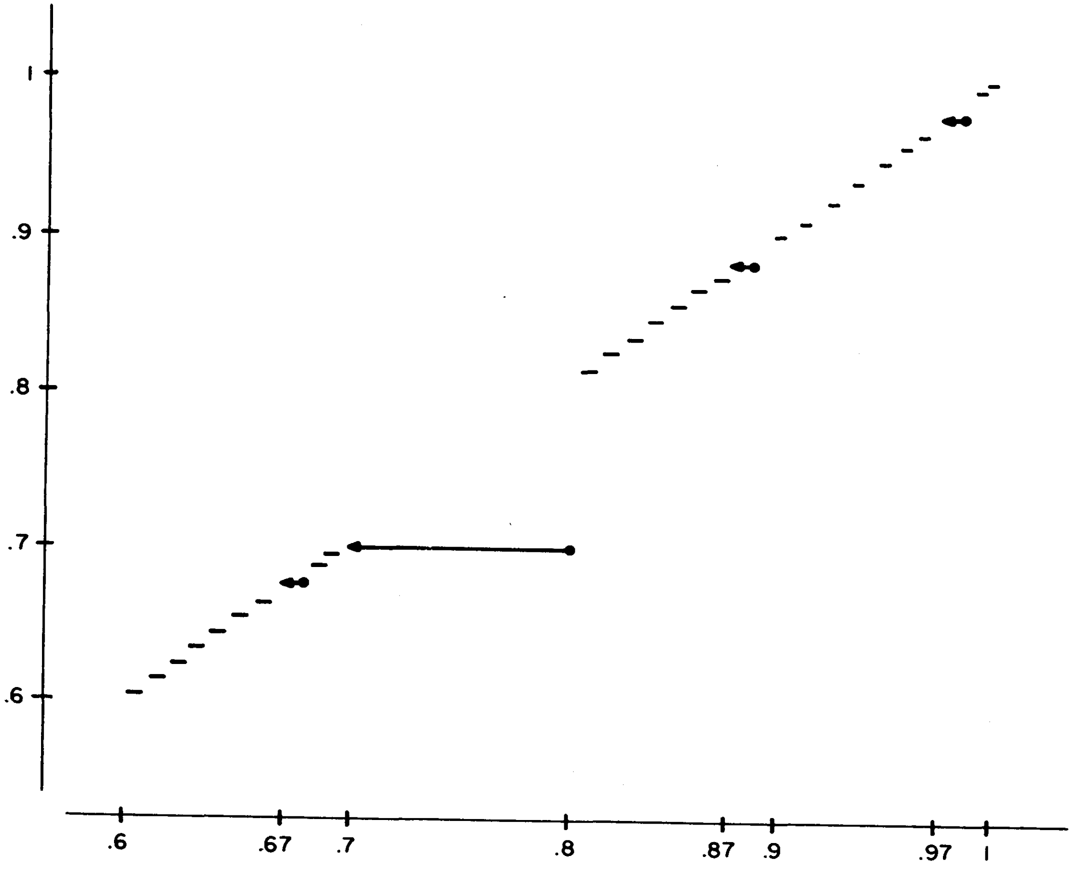
(iii) The graph of f contains points in every interval of each of the horizontal lines at distance 0, 1, 2,..., above the horizontal axis.



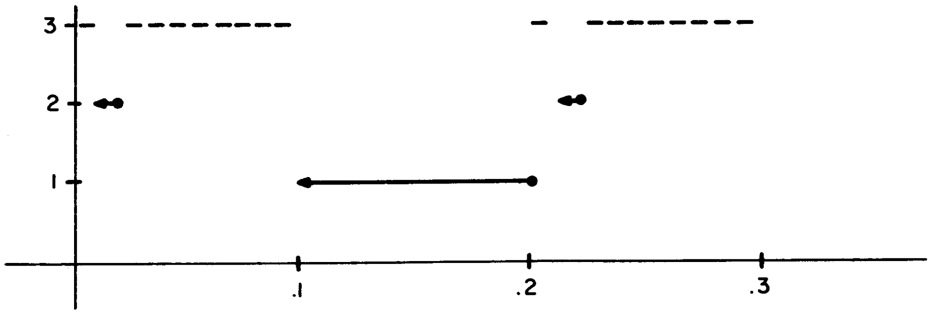
(iv) The graph of f contains points in every interval of the horizontal axis and of the horizontal line at distance 1 above the horizontal axis.



(v) The figure below shows a (rough) picture of the part of the graph of f which lies over [6/10, 1].



(vi) The figure below shows a (rough) picture of the graph of f. Notice that different scales have been used on the two axes.



18. See page 79 of the text.

20. (a) The first part is a straightforward computation. By Problem 1-17, the minimum of these numbers is

$$d^2 + c^2 - \frac{(-2md - 2c)^2}{4(m^2 + 1)} = \frac{4m^2d^2 + 4d^2 + 4m^2c^2 + 4c^2 - (4m^2d^2 + 8mcd + 4c^2)}{4(m^2 + 1)}$$

$$= \frac{d^2 + m^2c^2 - 2mcd}{m^2 + 1} = \frac{(cm - d)^2}{m^2 + 1}.$$

(b) The distance from (c,d) to the graph of f is the same as the distance from (c, d - b) to the graph of $g(x) = mx$. By part (a), this is

$$\frac{|cm - d + b|}{m^2 + 1}.$$

21. (a)

$x'$ = distance from $(x, y)$ to the graph of $f(x) = -x$ if $(x, y)$ lies above this graph (i.e., if $x + y \geq 0$), and the negative of this distance if $x + y \leq 0$.

$y'$ = distance from $(x, y)$ to the graph of $f(x) = x$ if $(x, y)$ lies above this graph (i.e., if $x - y < 0$), and the negative of this when $x - y \geq 0$.

By Problem 20, these distances are given by

$$\frac{|-x-y|}{\sqrt{2}} = \left|\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}\right|,$$

$$\frac{|x-y|}{\sqrt{2}} = \left|\frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}}\right|,$$

from which the desired formulas follow.

(b) Since

$$\frac{x'}{\sqrt{2}} = \frac{x}{2} + \frac{y}{2},$$

$$\frac{y'}{\sqrt{2}} = -\frac{x}{2} + \frac{y}{2},$$

we have $(x'/\sqrt{2})^2 - (y'/\sqrt{2})^2 = 1$ if and only if

$$\begin{aligned} 1 &= \left(\frac{x}{2} + \frac{y}{2}\right)^2 - \left(-\frac{x}{2} + \frac{y}{2}\right)^2 \\ &= \frac{x^2}{4} + \frac{y^2}{4} + \frac{xy}{2} - \left(\frac{x^2}{4} + \frac{y^2}{4} - \frac{xy}{2}\right) \\ &= xy. \end{aligned}$$

# CHAPTER 5

1. (i) $\lim_{x \to 1} \frac{x^2 - 1}{x + 1} = \frac{0}{2} = 0.$

   (iii) $\lim_{x \to 3} \frac{x^3 - 8}{x - 2} = \frac{27 - 8}{3 - 2} = 19.$

   (v) $\lim_{y \to x} \frac{x^n - y^n}{x - y} = \lim_{y \to x} \frac{y^n - x^n}{y - x} = ny^{n-1}.$

2. (iv) Let $\delta = \epsilon$, since $|x/(1 + \sin^2 x)| \le |x|$.

   (vi) If $\epsilon > 1$, let $\delta = 1$. Then $|x - 1| < \delta$ implies that $0 < x < 2$, so $0 < \sqrt{x} < 2$, so $|\sqrt{x} - 1| < 1$. If $\epsilon < 1$, then $(1 - \epsilon)^2 < x < (1 + \epsilon)^2$ implies that $|\sqrt{x} - 1| < \epsilon$, so it suffices to choose $\delta$ so that $(1 - \epsilon)^2 \le 1 - \delta$ and $1 + \delta \le (1 + \epsilon)^2$. Thus we can choose $\delta = 2\epsilon - \epsilon^2$.

3. (i), (ii), (iii) All numbers a which are not integers.

   (iv) All a.

   (v) All a with $a \neq 0$ and $a \neq 1/n$ for any integer n.

   (vi) All a with $|a| < 1$ and $a \neq 1/n$ for any integer n.

4. (a) (i) All a not of the form $n + k/10$ for integers n and k.

   (ii) All a not of the form $n + k/100$ for integers n and k.

   (iii), (iv) No a.

(v) All numbers a whose decimal expansion does not end 7999... .

(vi) All numbers a whose decimal expansion contains at least one 1.

(b) The answers are the same as in part (a) (although the description of the numbers in terms of their new "decimal expansions" may be different).

5. (ii) We need

$$|f(x) - 2| < \min\left(1, \frac{\epsilon}{2(|4| + 1)}\right) \quad \text{and} \quad |g(x) - 4| < \frac{\epsilon}{2(|2| + 1)},$$

so we need

$$0 < |x - 2| < \min\left(\sin^2\left(\frac{[\min(1, \epsilon/10)]^2}{9}\right) + \min(1, \epsilon/10), [\min(1, \epsilon/6)]^2\right) = \delta.$$

(iv) We need

$$\left|\frac{1}{g(x)} - \frac{1}{4}\right| < \frac{\epsilon}{2(|2| + 1)} \quad \text{and} \quad |f(x) - 2| < \min\left(1, \frac{\epsilon}{2(|1/4| + 1)}\right),$$

so we need

$$0 < |x - 2| < \min\left(\left[\min\left(2, \frac{8\epsilon}{2(|2| + 1)}\right)\right]^2, \sin^2\left(\frac{[\min(1, 2\epsilon/5)]^2}{9}\right) + \min(1, 2\epsilon/5)\right) = \delta.$$

6. Let $f(x) = \sqrt{|x|}$ with $a = 0$ and $\ell = 0$. Then for $\epsilon < 1$ we have $|\sqrt{|x|} - 0| < \epsilon$ when $0 < |x - 0| < \epsilon^2$; but if $0 < |x - 0| < \epsilon^2/2$, it does not follow that $|\sqrt{|x|} - 0| < \epsilon/2$ (instead we must let $0 < |x - 0| < (\epsilon/2)^2$.)

7. (a) Yes. For example, if $g = 1 - f$, then $\lim_{x\to a} [f(x) + g(x)]$ exists even if $\lim_{x\to a} f(x)$ (and consequently $\lim_{x\to a} g(x)$) does not exist; and if $g = 1/f$ where $f(x) \neq 0$ for all $x \neq a$, then $\lim_{x\to a} f(x)g(x)$ does exist even if $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ do not exist (for example, if $f(x) = 1/(x - a)$ for $x \neq 0$, and $g(x) = x - a$).

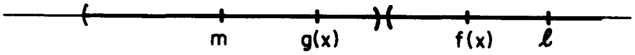
(b) Yes, since $g = (f + g) - f$.

(c) No. (This is just another way of stating part (b).)

(d) No. The argument analagous to part (b), that $g = (f\cdot g)/f$, will not work if $\lim_{x\to a} f(x) = 0$, and this is precisely the case in which one can find a counterexample. For example, let $f(x) = x - a$, and let $g(x) = 0$ for $x$ rational and 1 for $x$ irrational. Then $\lim_{x\to a} g(x)$ does not exist, but $\lim_{x\to a} f(x)g(x) = 0$, since $|f(x)g(x) - 0| \leq |f(x)|$.

10. Intuitively this is true because we only have to consider $x$'s satisfying $0 < |x - a| < \delta'$, where we can pick $\delta' < \delta$. In fact, if $\lim_{x\to a} f(x) = \ell$, and $\epsilon > 0$, there is a $\delta'$ such that if $0 < |x - a| < \delta'$, then $|f(x) - \ell| < \epsilon$. Now there is also a $\delta' < \delta$ with this property (namely, $\min(\delta', \delta)$). Since $f(x) = g(x)$ for all $x$ with $0 < |x - a| < \delta$, we also have $f(x) = g(x)$ for all $x$ with $0 < |x - a| < \delta'$, so the conclusion $|f(x) - \ell| < \epsilon$ can just as well be written $|g(x) - \ell| < \epsilon$. This shows that $\lim_{x\to a} g(x) = \ell$.

11. (a) Intuitively, $f(x)$ cannot be made close to a number $> \lim_{x \to a} g(x)$ because $f(x) \leq g(x)$ and $g(x)$ is close to $\lim_{x \to a} g(x)$. A rigorous proof is by contradiction. Suppose that $\ell = \lim_{x \to a} f(x) > \lim_{x \to a} g(x) = m$. Let $\epsilon = \ell - m > 0$. Then there is a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|\ell - f(x)| < \epsilon/2$ and $|m - g(x)| < \epsilon/2$.



Thus for $0 < |x - a| < \delta$ we have

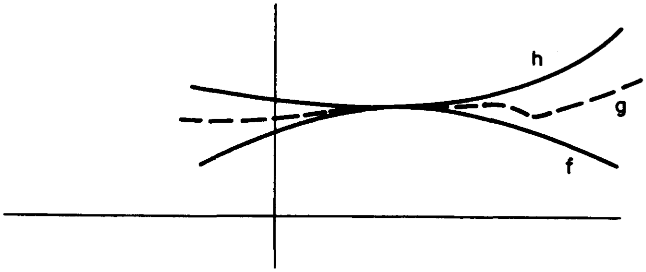
$$g(x) < m + \frac{\epsilon}{2} = \ell - \frac{\epsilon}{2} < f(x),$$

contradicting the hypothesis.

(b) It suffices to assume that $f(x) \leq g(x)$ for all $x$ satisfying $0 < |x - a| < \delta$, for some $\delta > 0$.

(c) No. For example, let $f(x) = 0$ and let $g(x) = |x|$ for $x \neq 0$, and $g(0) = 1$. Then $\lim_{x \to 0} f(x) = 0 = \lim_{x \to 0} g(x)$.

12. Intuitively, $g$ is squeezed between $f$ and $h$, which approach the same number:



Let $\ell = \lim_{x \to a} f(x)$. Given $\epsilon > 0$, there is $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|h(x) - \ell| < \epsilon$ and $|f(x) - \ell| < \epsilon$. Thus, if $0 < |x - a| < \delta$, then

$$\ell - \epsilon < f(x) \leq g(x) \leq h(x) < \ell + \epsilon,$$

so $|g(x) - \ell| < \epsilon$.

13. (a) We ought to have

$$\lim_{x \to 0} \frac{f(bx)}{x} = \lim_{x \to 0} \frac{bf(bx)}{bx} = b \lim_{x \to 0} \frac{f(bx)}{bx} = b \lim_{y \to 0} \frac{f(y)}{y} = b\ell.$$

The next to last equality can be justified as follows. If $\epsilon > 0$ there is a $\delta > 0$ such that if $0 < |y| < \delta$, then $|f(y)/y| < \epsilon$. Then if $0 < |x| < \epsilon/|b|$, we have $0 < |bx| < \epsilon$, so $|f(bx)/bx| < \epsilon$.

(b) In this case, $\lim_{x \to 0} f(bx)/x = \lim_{x \to 0} f(0)/x$ does not exist, unless $f(0) = 0$.

(c) Part (a) shows that $\lim_{x \to 0} (\sin 2x)/x = 2 \lim_{x \to 0} (\sin x)/x$. We can also use the following computation:

$$\lim_{x \to 0} \frac{\sin 2x}{x} = \lim_{x \to 0} \frac{2(\sin x)(\cos x)}{x} = 2 \lim_{x \to 0} \cos x \lim_{x \to 0} \frac{\sin x}{x} = 2.$$

(Of course this method won't work in general for $\lim_{x \to 0} (\sin \alpha x)/x$.)

14. (a) Intuitively, if $f(x)$ is close to $\ell$, then $|f(x)|$ is close to $|\ell|$. In fact, given $\epsilon > 0$ there is a $\delta > 0$ such that if $0 < |x - a| < \delta$, then

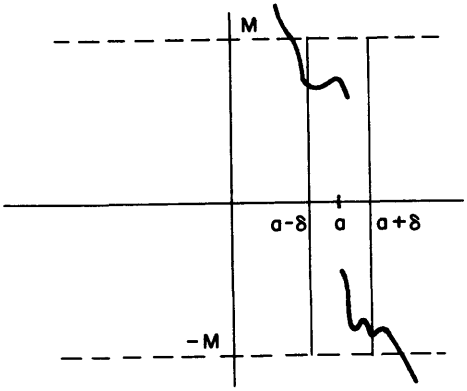
$|f(x) - \ell| < \epsilon$. But $||f(x)| - |\ell|| \le |f(x) - \ell| < \epsilon$ (by Problem 1-12(vi)).

(b) This follows from (a) and Theorem 2, since

$$\max(f, g) = \frac{f + g + |f - g|}{2},$$

$$\min(f, g) = \frac{f + g - |f - g|}{2}.$$

16. Pictorially, this means that f is bounded in an interval around a.

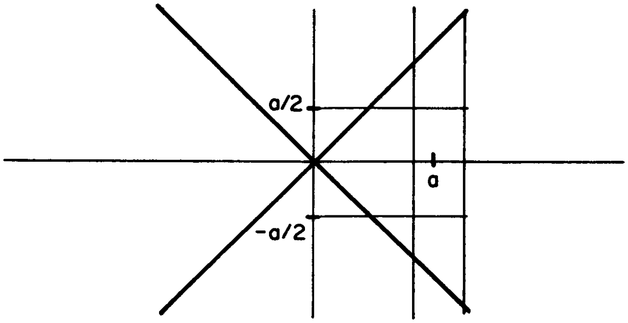


Choose $\delta > 0$ so that $|f(x) - \ell| < 1$ for $0 < |x - a| < \delta$ (we are picking $\epsilon = 1$). Then $\ell - 1 < f(x) < \ell + 1$, so we can let $M = \max(|\ell + 1|, |\ell - 1|)$.

17. For any $\delta > 0$ we have $f(x) = 0$ for some x satisfying $0 < |x - a| < \delta$ (namely, irrational x with $0 < |x - a| < \delta$) and also $f(x) = 1$ for some x satisfying $0 < |x - a| < \delta$ (namely, rational x with $0 < |x - a| < \delta$). This means that

we cannot have $|f(x) - \ell| < 1/2$ no matter what $\ell$ is. (There is a slight bit of cheating here; see Problem 8-5.)

18. Consider, for simplicity, the case $a > 0$. The basic idea is that since $f(x)$ is close to $a$ for all rational $x$ close to $a$, and close to $-a$ for all irrational $x$ close to $a$, we cannot have $f(x)$ close to any fixed number. To make this idea work, we note that for any $\delta > 0$ there are $x$ with $0 < |x - a| < \delta$ and $f(x) > a/2$ as well as $x$ with $0 < |x - a| < \delta$ and $f(x) < -a/2$. Since the distance between $a/2$ and $-a/2$ is $a$, this means that we cannot have $|f(x) - \ell| < a$ for all such $x$, no matter what $\ell$ is.



19. (a) follows from (b), since $|\sin 1/x| \le 1$ for all $x$ ($\neq 0$).

    (b) If $\delta > 0$ is such that $|g(x)| < \epsilon/M$ for all $x$ with $0 < |x| < \delta$, then $|g(x)h(x)| < \epsilon$ for all such $x$.

20. If $\lim_{x\to 0} f(x)$ does exist, then it is clear that $\lim_{x\to 0} [f(x) + g(x)]$ does not exist whenever $\lim_{x\to 0} g(x)$ does not exist (this was Problem 7(b) and (c)). On the other hand, if $\lim_{x\to 0} f(x)$ does not exist, choose $g = -f$; then $\lim_{x\to 0} g(x)$ does not exist, but $\lim_{x\to 0} [f(x) + g(x)]$ does exist.

21. (a) If $\lim_{x\to 0} f(x)g(x)$ existed, then $\lim_{x\to 0} g(x) = \lim_{x\to 0} f(x)g(x)/f(x)$ would also exist.

(b) Clearly, if $\lim_{x\to 0} f(x)g(x)$ exists, then $\lim_{x\to 0} g(x) = 0$.

(c) In case (1) of the hint, we clearly cannot have $\lim_{x\to 0} f(x) = 0$, so by assumption the limit does not exist at all. Let $g = 1/f$. Since it is not true that $\lim_{x\to 0} |f(x)| = \infty$, it follows that if $\lim_{x\to 0} g(x)$ exists, then $\lim_{x\to 0} g(x) \neq 0$. But this would imply that $\lim_{x\to 0} f(x)$ exists, so $\lim_{x\to 0} g(x)$ does not exist. On the other hand, $\lim_{x\to 0} f(x)g(x)$ clearly exists. In case (2), choose $x_n$ as in the hint. Define $g(x) = 0$ for $x \neq x_n$, and $g(x) = 1$ for $x = x_n$. Then $\lim_{x\to 0} g(x)$ does not exist, but $\lim_{x\to 0} f(x)g(x) = 0$.

22. Given $\epsilon > 0$, pick $n$ with $1/n < \epsilon$ and let $\delta$ be the minimum distance from $a$ to all points in $A_1,\ldots,A_n$ (except $a$ itself if $a$ is one of these points). Then $0 < |x - a| < \delta$ implies that $x$ is not in $A_1,\ldots,A_n$, so $f(x) = 0$ or $1/m$ for $m > n$, so $|f(x)| < \epsilon$.

24. (a) Although $\lim_{x\to 1} 1/x = 1$ is true, it is not true that for all $\delta > 0$ there is an $\epsilon > 0$ with $|1/x - 1| < \epsilon$ for $0 < |x - 1| < \delta$. In fact, if $\delta = 1$, there is no such $\epsilon$, since $1/x$ can be arbitrarily large for $0 < |x - 1| < 1$.

Moreover, any bounded function f automatically satisfies the condition, whether $\lim_{x\to a} f(x) = \ell$ is true or not.

(b) If f is a constant function, $f(x) = c$, this condition does not hold, since $|f(x) - c| < 1$ certainly does not imply that $0 < |x - a| < \delta$ for any $\delta$.

Moreover, the function $f(x) = x$, for example, satisfies this condition no matter what a and $\ell$ are.

25. (i), (ii), (iii), (iv) Both one-sided limits exist for all a.

(v) Both one-sided limits exist for $a \neq 0$, and neither exists for $a = 0$.

(vi) Both one-sided limits for all a with $|a| < 1$; moreover, $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$ exist.

26. (a) (i), (ii) Both one-sided limits exist for all a.

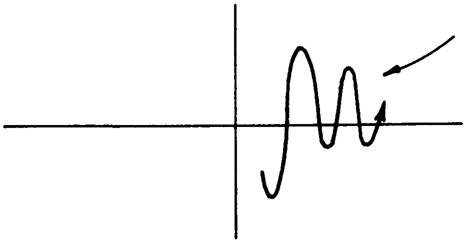
(iii), (iv) Neither one-sided limit exists for any a.

(v) Both one-sided limits exist for all a.

(vi) Both one-sided limits exist for all a whose decimal expansion contains at least one 1; in addition, the right-hand limit exists for all a whose decimal expansion contains no 1's, but which end in 0999... .

(b) The answers are the same as in part (a).

29.



Let $\ell = \lim_{x\to a^-} f(x)$ and $m = \lim_{x\to a^+} f(x)$. Since $m - \ell > 0$, there is a $\delta > 0$ so that

$$|f(x) - \ell| < \frac{m-\ell}{2} \quad \text{when} \quad a - \delta < x < a,$$

$$|f(y) - m| < \frac{m-\ell}{2} \quad \text{when} \quad a < y < a + \delta.$$

This implies that

$$f(x) < \ell + \frac{m-\ell}{2} = m - \frac{\ell - m}{2} < f(y).$$

The converse is false, as shown by $f(t) = t$ and any a. It is only possible to conclude that $\lim_{x\to a^-} f(x) \leq \lim_{x\to a^+} f(x)$.

30. Naturally we are assuming $a_n \neq 0$ and $b_n \neq 0$. If $x \neq 0$, then

$$\frac{a_n x^n + \dots + a_0}{b_m x^m + \dots + b_0} = \frac{a_n + \dfrac{a_{n-1}}{x} + \dots + \dfrac{a_0}{x^n}}{\dfrac{b_m}{x^{n-m}} + \dots + \dfrac{b_0}{x^n}} = \frac{f(x)}{g(x)}.$$

If $m < n$, then $\lim\limits_{x\to\infty} f(x) = a_n$ but $\lim\limits_{x\to\infty} g(x) = 0$. This implies that $\lim\limits_{x\to\infty} f(x)/g(x)$ does not exist (otherwise we would have $\lim\limits_{x\to\infty} f(x) = [\lim\limits_{x\to\infty} f(x)/g(x)]\cdot[\lim\limits_{x\to\infty} g(x)] = 0$.) If $m \geq n$, we write

$$\frac{a_n x^n + \dots + a_0}{b_m x^m + \dots + b_0} = \frac{\dfrac{a_n}{x^{m-n}} + \dots + \dfrac{a_0}{x^m}}{b_m + \dots + \dfrac{b_0}{x^m}} = \frac{f(x)}{g(x)}.$$

Then $\lim\limits_{x\to\infty} f(x) = 0$ if $m > n$, and $a_n$ if $m = n$, while $\lim\limits_{x\to\infty} g(x) = b_m$. So $\lim\limits_{x\to\infty} f(x)/g(x) = 0$ if $m > n$, and $a_n/b_m$ if $m = n$.

32. $\lim\limits_{x\to-\infty} f(x) = \ell$ means that for all $\epsilon > 0$ there is some $N$ such that $|f(x) - \ell| < \epsilon$ for $x < N$.

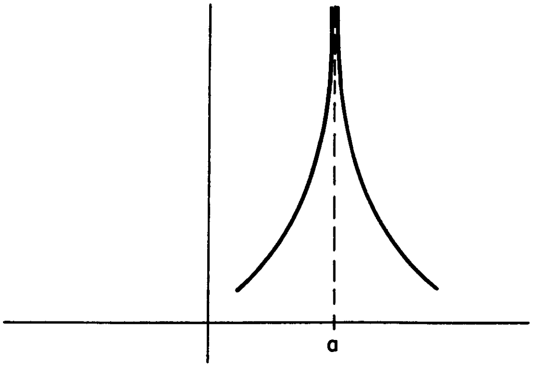
(a) The answer is the same as when $x \to \infty$ (Problem 30).

(b) If $\ell = \lim\limits_{x\to\infty} f(x)$, then for every $\epsilon > 0$ there is some $N$ such that

$|f(x) - \ell| < \epsilon$ for $x > N$. Now if $x < -N$, then $-x > N$, so $|f(-x) - \ell| < \epsilon$. So $\lim_{x \to -\infty} f(-x) = \ell$.

(c) If $\ell = \lim_{x \to -\infty} f(x)$, then for every $\epsilon > 0$ there is some $N$ such that $|f(x) - \ell| < \epsilon$ for $x < N$, and we can assume that $N < 0$. Now if $1/N < x < 0$, then $1/x < N$, so $|f(1/x) - \ell| < \epsilon$.

33.



(a) Given $N > 0$, let $\delta = 1/\sqrt{N}$. Then $0 < |x - 3| < \delta$ implies that $(x - 3)^2 < 1/N$, so $1/(x - 3)^2 > N$.

(b) Given $N > 0$, so that $1/N > 0$, choose $\delta > 0$ such that $|g(x)| < \epsilon/N$ for $0 < |x - a| < \delta$. Then $0 < |x - a| < \delta$ implies that $|f(x)/g(x)| > \epsilon \cdot (N/\epsilon) = N$.

34. (a) $\lim_{x\to a^+} f(x) = \infty$ means that for all N there is a $\delta > 0$ such that, for all x, if $a < x < a + \delta$, then $f(x) > N$.

$\lim_{x\to a^-} f(x) = \infty$ means that for all N there is a $\delta > 0$ such that, for all x, if $a - \delta < x < a$, then $f(x) > N$.

$\lim_{x\to\infty} f(x) = \infty$ means that for all N there is some M such that, for all x, if $x > M$, then $f(x) > N$.

It is also possible to define

$$\lim_{x\to-\infty} f(x) = \infty,$$

$$\lim_{x\to a} f(x) = -\infty,$$

$$\lim_{x\to a^+} f(x) = -\infty,$$

$$\lim_{x\to a^-} f(x) = -\infty,$$

$$\lim_{x\to\infty} f(x) = -\infty,$$

$$\lim_{x\to-\infty} f(x) = -\infty.$$

(b) Given $N > 0$, choose $\delta = 1/N$. If $0 < x < \delta$, then $1/x > N$.

(c) If $\lim_{x\to\infty} f(1/x) = \infty$, then for all N there is some M such that $f(1/x) > N$ for $x > M$. Choose $M > 0$. If $0 < x < 1/M$, then $x > M$, so $f(x) > N$. Thus $\lim_{x\to 0^+} f(x) = \infty$. The reverse direction is similar.

# CHAPTER 6

1. (ii) No F, since $\lim_{x \to 0} |x|/x$ does not exist.

   (iv) No F, since F(a) would have to be 0 for irrational a, and then F is not continuous at a if a is rational.

2. Problem 4-15:

   (i), (ii), (iii) All points except integers.

   (iv) All points.

   (v) All points except 0 and 1/n for integers n.

   (vi) All points in (-1, 1) except 0 (where it is not defined) and 1/n for integers n.

   Problem 4-17:

   (i) All points not of the form n + k/10 for integers k and n.

   (ii) All points not of the form n + k/100 for integers k and n.

   (iii), (iv) No points.

   (v) All points whose decimal expansion does not end 7999... .

   (vi) All points whose decimal expansion contains at least one 1.

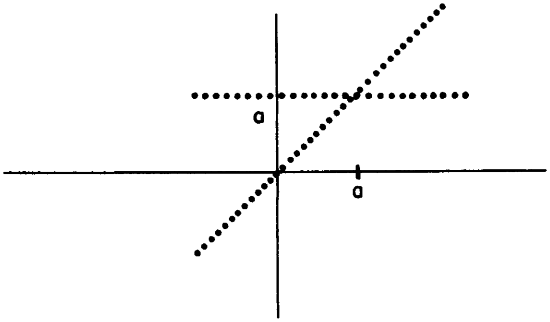
3. (a) Clearly $\lim_{h \to 0} f(h) = 0$, since $|h| < \delta$ implies that $|f(h) - f(0)| = |f(h)| < \delta$.

(b) Let $f(x) = 0$ for $x$ irrational, and $f(x) = x$ for $x$ rational.

(c) Notice that $|f(0)| \le |g(0)| = 0$, so $f(0) = 0$. Since $g$ is continuous at $0$, for every $\epsilon > 0$ there is a $\delta > 0$ such that $|g(h) - g(0)| = |g(h)| < \epsilon$ for $|h| < \delta$. Thus, if $|h| < \delta$, then $|f(h) - f(0)| = |f(h)| \le |g(h)| < \epsilon$. So $\lim_{h \to 0} f(h) = 0 = f(0)$.

4. Let $f(x) = 1$ for $x$ rational, and $f(x) = -1$ for $x$ irrational.

5. Let $f(x) = a$ for $x$ irrational, and $f(x) = x$ for $x$ rational.



6. (a) Define $f$ as follows (see the solution to Problem 4-15(vi)):

$$f(x) = \begin{cases} 0, & x \le 0 \\ \dfrac{1}{[\frac{1}{x}]}, & 0 < x \le 1 \\ 2, & x > 1. \end{cases}$$

(b) Let

$$f(x) = \begin{cases} -1, & x \le 0 \\ \dfrac{1}{[\frac{1}{x}]}, & 0 < x \le 1 \\ 2, & x > 1. \end{cases}$$

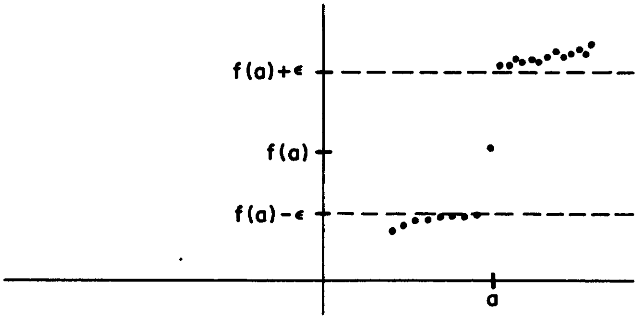
7. Note that $f(x + 0) = f(x) + f(0)$, so $f(0) = 0$. Now

$$\begin{aligned} \lim_{h\to 0} f(a + h) - f(a) &= \lim_{h\to 0} f(a) + f(h) - f(a) \\ &= \lim_{h\to 0} f(h) \\ &= \lim_{h\to 0} f(h) - f(0) = 0, \end{aligned}$$

since f is continuous at 0.

8. Since $(f + \alpha)(a) \neq 0$, Theorem 3 implies that $f + \alpha$ is non-zero in some open interval containing a.

9. (a) This is just a restatement of the definition: If the condition did not hold, then for every $\epsilon > 0$ we would have $|f(x) - f(a)| \le \epsilon < 2\epsilon$ for all x sufficiently close to a, i.e., for all x satisfying $|x - a| < \delta$ for some $\delta > 0$. If this were true for all $\epsilon$, then f would be continuous at a.



(b) If neither of these conditions held, then for every $\epsilon > 0$ there would be $\delta_1, \delta_2 > 0$ such that $f(x) \geq f(a) - \epsilon$ for $|x - a| < \delta_1$ and $f(x) \leq f(a) + \epsilon$ for $|x - a| < \delta_2$. If $|x - a| < \delta = \min(\delta_1, \delta_2)$, then $f(a) - \epsilon \leq f(x) \leq f(a) + \epsilon$, so $|f(x) - f(a)| \leq \epsilon$. Since this would be true for all $\epsilon > 0$, it would follow that f is continuous at a.

10. (a)

$$\lim_{x \to a} |f|(x) = \left|\lim_{x \to a} f(x)\right| \qquad \text{by Problem 4-14}$$

$$= |f(a)| = |f|(a).$$

(b) The formulas for E and O in the solution to Problem 3-13 show that E and O are continuous if f is.

(c) This follows from part (a), since

$$\max(f, g) = \frac{f + g + |f - g|}{2},$$

$$\min(f, g) = \frac{f + g - |f - g|}{2}.$$

(d) Let $g = \max(f, 0)$ and $h = -\min(f, 0)$.

11. $1/g = f \circ g$, and $f$ is continuous at $g(a)$ if $g(a) \neq 0$. So by Theorem 2, $1/g$ is continuous at $a$ if $g(a) \neq 0$.

12. (a) Clearly $G$ is continuous at $a$, since $G(a) = \ell = \lim_{x \to a} g(x) = \lim_{x \to a} G(x)$. So $f \circ G$ is continuous at $a$ by Theorem 2. Thus

$$f(\ell) = f(G(a)) = (f \circ G)(a) = \lim_{x \to a} (f \circ G)(x) = \lim_{x \to a} f(g(x)).$$

(b) Let $g(x) = \ell + x - a$ and

$$f(x) = \begin{cases} 0, & x \neq \ell \\ 1, & x = \ell. \end{cases}$$

Then $\lim_{x \to a} g(x) = \ell$, so $f(\lim_{x \to a} g(x)) = f(\ell) = 1$; but $g(x) \neq \ell$ for $x \neq a$, so $\lim_{x \to a} f(g(x)) = \lim_{x \to a} 0 = 0$.

13. (a) Since $f$ is continuous on $[a, b]$, the limits $\lim_{t \to a^+} f(t)$ and $\lim_{t \to b^-} f(t)$ exist. Let

$$g(x) = \begin{cases} \lim\limits_{t \to a^+} f(t), & x \leq a \\ f(x), & a \leq x \leq b \\ \lim\limits_{t \to b^-} f(t), & b \leq x. \end{cases}$$

(b) Let $f(x) = 1/(x - a)$.

14. (a) The limit $\lim\limits_{t \to 0} f(a + t)$ exists, and equals $f(a) = g(a) = h(a)$, since

$$\lim_{t \to 0^+} f(a + t) = \lim_{t \to 0^+} g(a + t) = g(a),$$

$$\lim_{t \to 0^-} f(a + t) = \lim_{t \to 0^-} h(a + t) = h(a).$$

(b) $f$ is continuous at $c$ by (a), and at any $x \neq c$, in $[a, b]$, since $f$ agrees with either $g$ or $h$ on some interval around $x$.

15. If $f$ is continuous on $[a, b]$ and $f(a) > 0$, then there is some $\delta > 0$ such that for all $x$, if $a \leq x < x + \delta$, then $|f(x) - f(a)| < f(a)$. This last inequality implies that $f(x) > 0$. The proof for $f(b) > 0$ is similar.

16. (a) No in the first case; yes in the second.

(b) We have

$$\begin{aligned} \lim_{x \to a} g(x) &= \lim_{x \to a} f(x), && \text{since } g(x) = f(x) \text{ for } x \neq a \\ &= g(a), && \text{by definition of } g(a). \end{aligned}$$

(c) $g(x) = 0$ for all $x$.

(d) Since $g(a) = \lim_{y \to a} f(y)$, by definition, it follows that for any $\epsilon > 0$ there is a $\delta > 0$ such that $|f(y) - g(a)| < \epsilon$ for $|y - a| < \delta$. This means that

$$g(a) - \epsilon < f(y) < g(a) + \epsilon$$

for $|y - a| < \delta$. So if $|x - a| < \delta$, we have

$$g(a) - \epsilon \leq \lim_{y \to x} f(y) \leq g(a) + \epsilon,$$

which shows that $|g(x) - g(a)| \leq \epsilon$ for all $x$ satisfying $|x - a| < \delta$. Thus $g$ is continuous at $a$.

# CHAPTER 7

1. (ii) Bounded above and below; no maximum or minimum value.

(iv) Bounded below but not above; minimum value 0.

(v) Bounded above and below. It is understood that $a > -1$ (so that $-a - 1 < a + 1$). If $-1 < a \leq 1/2$, then $a < -a - 1$, so $f(x) = a + 2$ for all $x$ in $(-a - 1, a + 1)$, so $a + 2$ is the maximum and minimum value. If $-1/2 < a \leq 0$, then $f$ has the minimum value $a^2$, and if $a \geq 0$, then $f$ has the minimum value 0. Since $a + 2 > (a + 1)^2$ only for $[-1 - \sqrt{5}]/2 < a < [1 + \sqrt{5}]/2$, when $a \geq -1/2$ this function $f$ has a maximum value only for $a \leq [1 + \sqrt{5}]/2$ (the maximum value being $a + 2$).

(vi) Bounded above and below. As in part (v), it is assumed that $a > -1$. If $a \leq -1/2$ then $f$ has the minimum and maximum value $3/2$. If $a \geq 0$, then $f$ has the minimum value 0, and the maximum value $\max(a^2, a + 2)$. If $-1/2 < a < 0$, then $f$ has the maximum value $3/2$ and no minimum value.

(viii) Bounded above and below; maximum value 1; no minimum value.

(x) Bounded above and below; minimum value 0; the maximum value is $a$ if $a$ is rational, and there is no maximum value if $a$ is irrational.

(xii) Bounded above and below; minimum value 0; maximum value $[a]$.

2. (ii) $n = -5$, since $f(-5) = 2(-5) + 1 < 0 < f(-4)$.

(iv) $n = 0$, since both roots of $f(x) = 0$ lie in $[0, 1]$.

3. (ii) If $f(x) = \sin x - x + 1$, then $f(0) > 0$ and $f(2) = (\sin 2) - 1 < 0$.

4. (a) Let $\ell = (n - k)/2$ and let

$$f(x) = (x^{2\ell} + 1)(x - 1)(x - 2)\cdot\ldots\cdot(x - k).$$

(b) If $f$ has roots $a_1,\ldots,a_r$ with multiplicities $m_1,\ldots,m_r$, so that $k = m_1 + \ldots + m_r$, then

$$f(x) = (x - a_1)^{m_1}\cdot\ldots\cdot(x - a_r)^{m_r}\,g(x)$$

where $g$ is a polynomial function of degree $n - (m_1 + \ldots + m_r) = n - k$ with no roots. It follows from Theorem 9 that $n - k$ is even.

6. If not, then $f$ takes on both positive and negative values, so $f$ would have the value 0 somewhere in $(-1, 1)$, which is impossible, since $\sqrt{1 - x^2} \neq 0$ for $x$ in $(-1, 1)$.

8. If not, then $f(x) = g(x)$ for some $x$ and $f(y) = -g(y)$ for some $y$. But $f$ is either always positive or always negative, since $f(x) \neq 0$ for all $x$. So $g(x)$ and $g(y)$ have different signs. This implies that $g(z) = 0$ for some $z$, which is impossible, since $0 \neq f(z) = \pm g(z)$.

9. (a) $f(x) > 0$ for all $x \neq a$. For if $x_0 > a$ is the point with $f(x_0) > 0$, and if $f(x) < 0$ for some $x > a$, then $f(z) = 0$ for some $z$ in the interval between $x_0$ and $x$; since $z \neq a$, this contradicts the hypothesis. The proof for $x < a$ is similar.

(b) For $y \neq 0$, let $f(x) = x^2 + xy + y^2$ (to be very explicit, we could use $f_y$ instead of $f$). Then $f(x) \neq 0$ for $x \neq 0$ and $f(x) > 0$ for $x = \pm y$, so $f(x) > 0$ for all $x \neq 0$ by part (a).

(c) Let $f(x) = x^3 + x^2y + xy^2 + y^3$, where $y \neq 0$. Then $f(x) \neq 0$ for $x \neq 0$. Suppose $y > 0$. Then $f(y) > 0$. This implies that $f(x) > 0$ for all $x > 0$; for if $f(x) < 0$, then $f(z) = 0$ for some $z$ in the interval between $x$ and $y$, which is impossible since $z \neq 0$. Moreover $f(-y) < 0$, and the same sort of argument shows that $f(x) < 0$ for all $x < 0$. On the other hand, if $y < 0$, then $f(x) < 0$ for $x > 0$ and $f(x) > 0$ for $x < 0$.

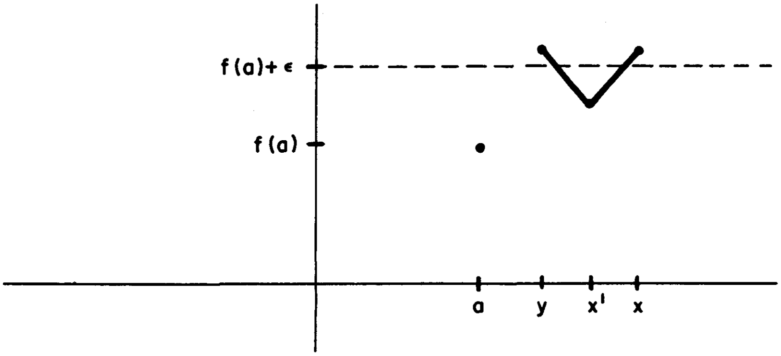
12. (a) Use the proof in the solution to Problem 11, but applied to $f$ and $-I$.

(b) Apply the same proof to $f$ and $g$.

13. (a) No, $f$ is not continuous on $[-1, 1]$.

If $a < b$ are two points in $[-1, 1]$ with $a,b > 0$ or $a,b < 0$, then $f$ takes every value between $f(a)$ and $f(b)$ on the interval $[a, b]$ since $f$ is continuous on [a, b]. On the other hand, if $a < 0 < b$, then f takes on all values between -1 and 1 on [a, b], so f certainly takes on all values between f(a) and f(b). The same argument works for $a = 0$ or $b = 0$ (because f(0) was defined to be in [-1, 1]).

(b) If f were not continuous at a, then (by Problem 6-9(b)) for some $\epsilon > 0$ there would be x arbitrarily close to a with $f(x) > f(a) + \epsilon$ or $f(x) < f(a) - \epsilon$, say the first. We can even assume that there are such x's arbitrarily close to a and $> a$, or else arbitrarily close to a and $< a$, say the first. Pick some $x > a$ with $f(x) > f(a) + \epsilon$. By the Intermediate Value Theorem, there is x' between a and x with $f(x') < f(a) + \epsilon$. But there is also y between a and x' with $f(y) > f(a) + \epsilon$. By the Intermediate Value Theorem, f takes on the value $f(a) + \epsilon$ between x and x' and also between x' and y, contradicting the hypothesis.



(c) As in (b), choose $x_1 > a$ with $f(x_1) > f(a) + \epsilon$. Then choose $x_1'$ between a and $x_1$ with $f(x_1') < f(a) + \epsilon$. Then choose $x_2$ between a and $x_1'$ with $f(x_2) > f(a) + \epsilon$ and $x_2'$ between a and $x_2$ with $f(x_2') < f(a) + \epsilon$. Etc. Then f takes on the value $f(a) + \epsilon$ on each interval $[x_n', x_n]$, contradicting the hypothesis.

14. (a) This is obvious since $|cf|(x) = |c| \cdot |f(x)|$ for all x in $[0, 1]$.

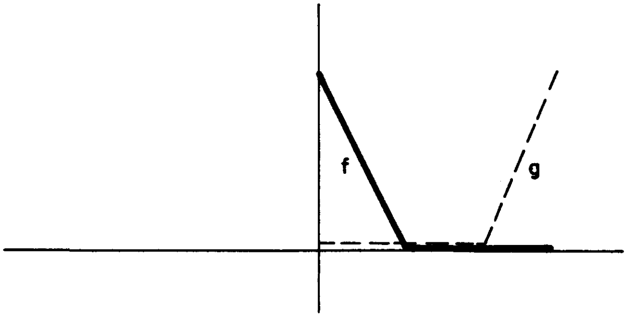
(b) We have

$$|f + g|(x) = |f(x) + g(x)| \leq |f(x)| + |g(x)| \leq |f|(x) + |g|(x).$$

If $|f + g|$ has its maximum value at $x_0$, then

$$||f + g|| = |f + g|(x_0) \leq |f|(x_0) + |g|(x_0) \leq ||f|| + ||g||.$$

If f and g are the two functions shown below, then $||f|| = ||g|| = ||f + g|| = 1$, so $||f + g|| \neq ||f|| + ||g||$. (Notice that this happens even though we have $|f + g|(x) = |f|(x) + |g|(x)$ for all x.)



(c) Apply part (b) with $f$ replaced by $h - g$ and $g$ replaced by $g - f$.

15. (a) Choose $b > 0$ so that $|\phi(b)/b^2| < 1/2$. Then

$$b^n + \phi(b) = b^n(1 + \frac{\phi(b)}{b^n}) > \frac{b^n}{2} > 0.$$

Similarly, if $a < 0$ and $|\phi(a)/a^2| < 1/2$, then $a^n + \phi(a) < a^n/2 < 0$. So $x^n + \phi(x) = 0$ for some $x$ in $[a, b]$.

(b) Choose $a > 0$ such that $a^n > 2\phi(0)$ and such that $|\phi(x)/x^n| < 1/2$ for $|x| > a$. Then for $|x| > a$ we have

$$x^n + \phi(x) = x^n(1 + \frac{\phi(x)}{x^n} > \frac{x^n}{2} > \frac{a^n}{2} > \phi(0),$$

so the minimum of $x^n + \phi(x)$ for $x$ in $[-a, a]$ is the minimum for all $x$.

16. If

$$f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0,$$

let

$$M = \max(1, 2n|a_{n-1}|, \dots, 2n|a_0|).$$

Then for all $x$ with $|x| \geq M$ we have

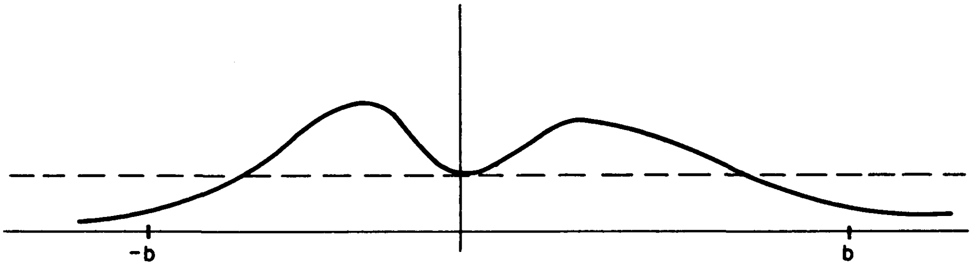
$$\frac{1}{2} \leq 1 + \frac{a_{n-1}}{x} + \dots + \frac{a_0}{x^n},$$

so

$$|f(x)| = |x^n(1 + \frac{a_{n-1}}{x} + \dots + \frac{a_0}{x^n})| \geq x^n/2 .$$

If $b > M$ satisfies $|b^n| \geq 2f(0)$, then $|f(x)| \geq |f(0)|$ for $|x| \geq b$. So the minimum value of $|f(x)|$ on $[-b, b]$ is the minimum value on $\mathbf{R}$. (Naturally this problem can be generalized exactly as in Problem 15: If $\phi$ is continuous and $\lim\limits_{x\to\infty} \phi(x)/x^n = 0 = \lim\limits_{x\to-\infty} \phi(x)/x^n$, then there is some number $y$ such that $|y^n + \phi(y)| \leq |x^n + \phi(x)|$ for all $x$.)

17. Pick $b > 0$ so that $f(x) < f(0)$ for $|x| > b$. Then the maximum of $f$ on $[-b, b]$ is also the maximum on $\mathbf{R}$.



18. (a) Apply Theorem 3 to the (continuous) function

$$d(z) = \sqrt{(f(z))^2 + (z - x)^2},$$

which gives the distance from $(x, 0)$ to $(z, f(z))$, for $z$ in $[a, b]$.

(b) If $f(x) = x$ on $(a, b)$, then no point of the graph is nearest to the point $(a, a)$.

(c) Clearly the function d of part (a) satisfies $\lim_{z \to \infty} d(z) = \infty = \lim_{z \to -\infty} d(z)$, since $d(z) \geq |z - x|$. Choose $c > 0$ so that $d(z) > d(0)$ for $|z| > c$. Then the minimum of d on $[-c, c]$ will be the minimum of d on $\mathbf{R}$ .

(d) By definition, $g(x) = \sqrt{(f(z))^2 + (z - x)^2}$ for some z in $[a, b]$. Now

$$\sqrt{(f(z))^2 + (z - y)^2} \leq \sqrt{(f(z))^2 + (z - x)^2} + |z - y| \qquad \text{for all } z.$$

So $g(y)$, the minimum of all $\sqrt{(f(z))^2 + (z - y)^2}$, is less than or equal to $|z - y|$ + the minimum of all $\sqrt{(f(z))^2 + (z - x)^2}$, which is $g(x) + |y - x|$. Since $|g(y) - g(x)| \leq |y - x|$ it follows that g is continuous (given $\epsilon > 0$, let $\delta = \epsilon$).

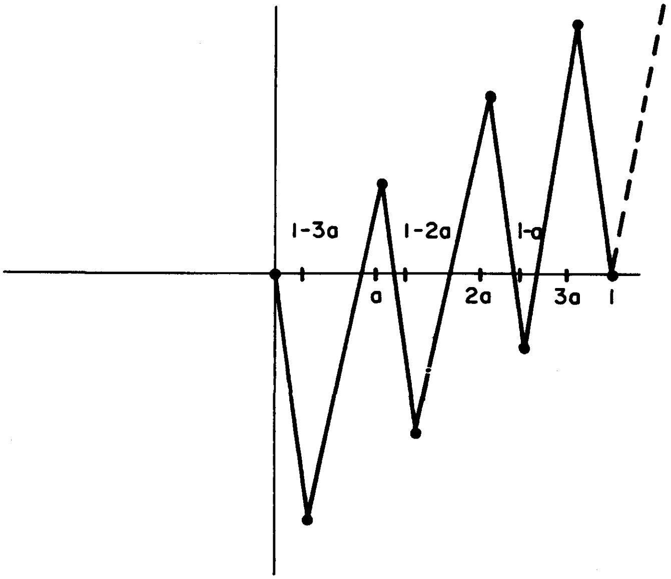
(e) Apply Theorem 3 to the continuous function g on $[a, b]$.

19. (a) If the continuous function g satisfied $g(x) \neq 0$ for all x, then either $g(x) > 0$ for all x or $g(x) < 0$ for all x, i.e., either $f(x) > f(x + 1/n)$ or $f(x) < f(x + 1/n)$ for all x. In the first case, for example, we would have

$$f(0) > f(\tfrac{1}{n}) > f(\tfrac{2}{n}) > \dots > f(\tfrac{n}{n}) = f(1),$$

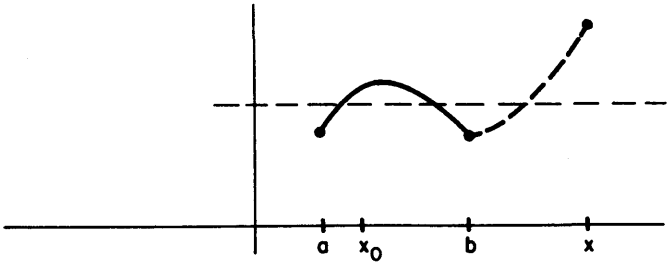
contradicting the hypothesis that $f(0) = f(1)$.

(b) The picture below illustrates such a function f when $1/4 < a < 1/3$.



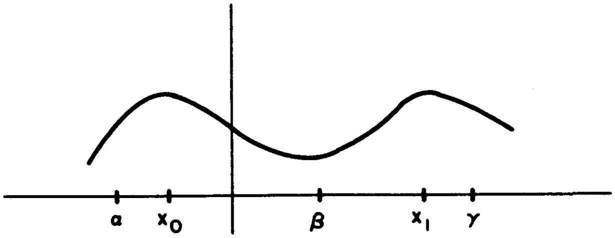
In general, if $1/(n + 1) < a < 1/n$, define f arbitrarily on $[0, a]$, subject only to the condition that $f(0) = 0$, $f(a) > 0$, and $f(1 - na) = -nf(a)$. Since $1/(n + 1) < a < 1/n$, the numbers 0, $1 - na$, and a are all distinct, so this is possible. Then define f on $[ka, (k + 1)a]$ by $f(ka + x) = f(x) + ka$. In particular, we have $f(1) = f(na + (1 - na)) = na + f(1 - na) = 0$, but $f(x + a) - f(x) = f(a) > 0$ for all x.

20. (a) If $f(a) = f(b)$ for $a < b$, then we cannot have $f(x_1) > f(a)$ and $f(x_2) < f(a)$ for some $x_1$, $x_2$ in $[a, b]$, since this would imply that $f(x) = f(a)$ for some x between $x_1$ and $x_2$, so that f would take on the value $f(a)$ three times. So either $f(x) > f(a)$ for all x in $(a, b)$, or else $f(x) < f(a)$ for all x in (a, b), say the first. Pick any $x_0$ in (a, b). The Intermediate Value Theorem implies that f takes on all values between f(a) and $f(x_0)$ in the interval $[a, x_0]$ and also in the interval $[x_0, b]$. So we cannot have $f(x) > f(a)$ for $x < a$ or $x > b$, since this would imply that f takes on these values yet a third time (on [x, a] or [b, x]).



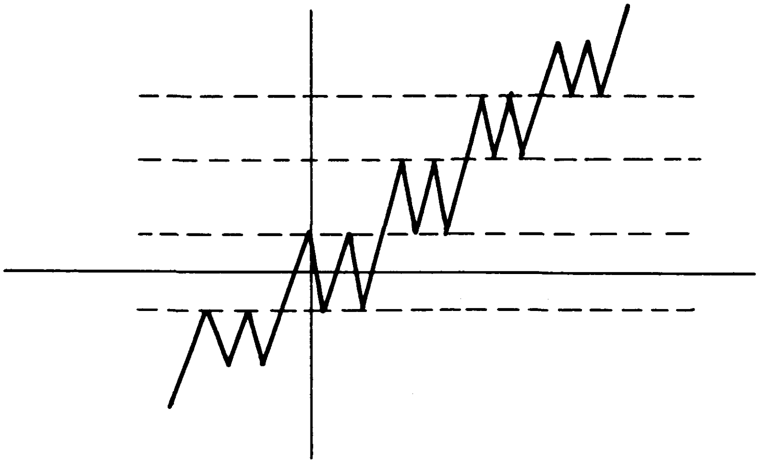
So f is actually bounded above on **R** (since it is bounded on [a, b]), which means that f does not take on every value.

(b) Moreover, even if we allowed the situation where f did not take on all values, it would still be true that f actually has a maximum value M on **R** (the maximum on [a, b] will be the maximum on **R** ). Now f must take on this maximum value twice, say at $x_0$ and $x_1$. Pick $\alpha < x_0 < \beta < y_0 < \gamma$.



If m is the maximum of $f(\alpha)$, $f(\beta)$, and $f(\gamma)$, then f takes on all values between m and M on each interval $[\alpha, x_0]$, $[x_0, \beta]$, $[\beta, x_1]$ and $[x_1, \gamma]$, which is impossible.

(c) The following picture, for $n = 5$, will indicate the general case.



(d) Pick $x_1 < \dots < x_n$ with $f(x_1) = \dots = f(x_n) = a$. In each interval $(x_i, x_{i+1})$, either $f > a$ or $f < a$. Since n is even, there are an odd number, $n - 1$, or such intervals, so either $f > a$ in more than half of them, or $f < a$ in more than half of them. Thus $f > a$ in at least $n/2$ of them, or $f < a$ in at least $n/2$ of them, say the first. Then f takes on all values slightly larger than a at least twice in at least $n/2$ intervals. This shows that f cannot take on these values any where else, so f is bounded above. (Moreover, the same sort of argument as in part (c) shows that f would have to take on values slightly less than the maximum value at least 2n times.)

## CHAPTER 8

1. (ii) 1 is the greatest element and -1 is the least element.

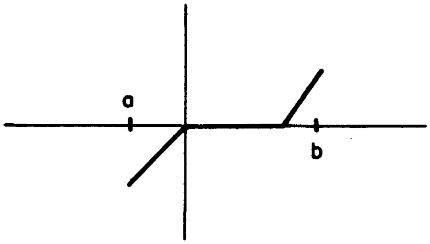
(iv) 0 is the least element, and the least upper bound is $\sqrt{2}$, which is not in the set.

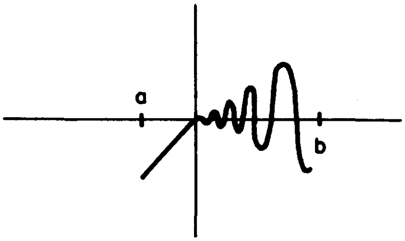
(vi) Since $\{x: x^2 + x + 1 < 0\} = ([-1 - \sqrt{5}]/2, [-1 + \sqrt{5}]/2)$, the greatest lower bound is $[-1 - \sqrt{5}]/2$ and the least upper bound is $[-1 + \sqrt{5}]/2$; neither belongs to the set.

(viii) $1 - 1/2$ is the greatest element, and the greatest lower bound is -1, which is not in the set.

2. (b) Since A is bounded below, $B \neq \emptyset$. Since $A \neq \emptyset$, there is some x in A. Then any $y > x$ is not an upper bound for A, so no such y is in B, so B is bounded above. Let $\alpha = \sup B$. Then $\alpha$ is automatically $\geq$ any lower bound for A, so it suffices to prove that $\alpha$ is a lower bound for A. Now if $\alpha$ were not a lower bound for A, then there would be some x in A with $x < \alpha$. Since $\alpha$ is the least upper bound of B, this would mean that there is some y in B with $x < y < \alpha$. But this is impossible, since $x < y$ means that y is not a lower bound for A, so y would not be in B.

3. (a) No. For example, the functions f shown below have no second smallest x with $f(x) = 0$.



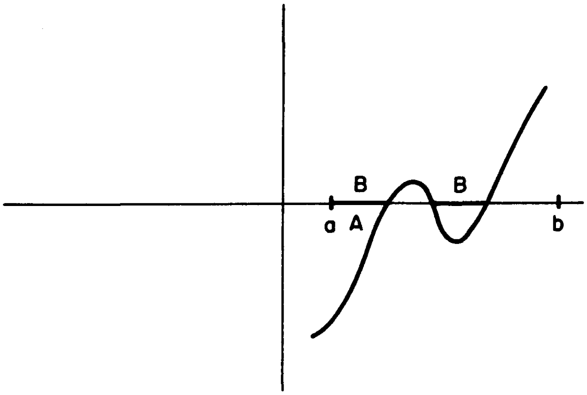


Since $b - a + x$ varies between $b$ and $a$ as $x$ varies between $a$ and $b$, the function $g(x) = f(b - a + x)$ satisfies $g(a) = f(b) > 0$ and $g(b) = f(a) < 0$. So there is a smallest $y$ with $g(y) = 0$. Then $x = b - a + y$ is the largest $x$ with $f(x) = 0$.

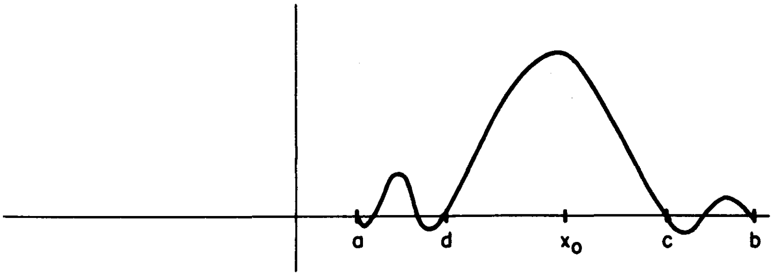
(b) Clearly $B \neq \emptyset$, since $a$ is in $B$; in fact, there is some $\delta > 0$ such that $B$ contains all points $x$ satisfying $a \leq x < a + \delta$, by Problem 6-15, since $f$ is continuous on $[a, b]$ and $f(a) < 0$. Similarly, $b$ is an upper bound for $B$, and, in fact, there is a $\delta > 0$ such that all points $x$ satisfying $b - \delta < x \leq b$ are upper bounds for $A$; this also follows from Problem 6-15, since $f$ is continuous on $[a, b]$ and $f(b) > 0$.

Let $\alpha = \sup A$. Then $a < \alpha < b$. Suppose $f(\alpha) < 0$. By Theorem 6-3, there is a $\delta > 0$ such that $f(x) < 0$ for $\alpha - \delta < x < \alpha + \delta$. This would mean that $\alpha + \delta/2$ is in $A$, a contradiction. Similarly, suppose $f(\alpha) > 0$. Then there is a $\delta > 0$ such that $f(x) > 0$ for $\alpha - \delta < x < \alpha + \delta$. But then $\alpha - \delta/2$ would also be an upper bound for $B$, contradicting the fact that $\alpha$ is the least upper bound. So $f(\alpha) = 0$.

This $\alpha$ is the greatest $x$ in $[a, b]$ with $f(x) = 0$. The sets A and B are different for the function shown below.



4. Let $c$ be the largest $x$ in $[a, x_0]$ with $f(x) = 0$ and $d$ the smallest $x$ in $[x_0, b]$ with $f(x) = 0$.



6. (a) By definition of continuity, we have $f(a) = \lim\limits_{x \to a} f(x)$ for all $a$, so it suffices to prove that $\lim\limits_{x \to a} f(x) = 0$ (knowing that the limit $\ell$ exists). Now given $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - \ell| < \epsilon$ for all $x$ satisfying $0 < |x - a| < \delta$. Since A is dense, there is a number $x$ in A satisfying $0 < |x - a| < \delta$; so $|0 - \ell| < \epsilon$. Since this is true for all $\epsilon > 0$, it follows that $\ell = 0$.

(b) Apply part (a) to $f - g$.

(c) As in part (b), it obviously suffices to show that if $f$ is continuous and $f(x) \geq 0$ for all numbers $x$ in $A$, then $f(x) \geq 0$ for all $x$. Now there is a $\delta > 0$ such that, for all $x$, if $0 < |x - a| < \delta$, then $|f(x) - \ell| < |\ell|/2$. This implies that $f(x) < \ell + |\ell|/2$; if $\ell < 0$, it would follow that $f(x) < 0$, which would be false for those $x$ in $A$ which satisfy $0 < |x - a| < \delta$.

It is not possible to replace $\geq$ by $>$ throughout. For example, if $f(x) = |x|$, then $f(x) > 0$ for all $x$ in the dense set $\{x: x \neq 0\}$, but it is not true that $f(x) > 0$ for all $x$.

7. According to Problem 3-16, we have $f(x) = cx$ for all rational $x$ (where $c = f(1)$). Since $f$ is continuous, it follows from Problem 6 that $f(x) = cx$ for all $x$ (apply Problem 6 to $f$ and $g(x) = cx$).

8. (a) The set $\{f(x): x < a\}$ is bounded above (by $f(a)$); let $\alpha = \sup\{f(x): x < a\}$. Then $\lim_{x \to a^-} f(x) = \alpha$: Given any $\epsilon > 0$, there is some $f(x)$ for $x < a$ with $f(x) > \alpha - \epsilon$, since $\alpha$ is the least upper bound of $\{f(x): x < a\}$. Let $\delta = a - x$. If $a - \delta < y < a$, then $x < y < a$, so $f(x) \leq f(y)$. This means that $\alpha \geq f(y) > \alpha - \epsilon$, so surely $|f(y) - \alpha| < \epsilon$.

The proof that $\lim_{x \to a^+} f(x) = \inf\{f(x) : x > a\}$ is similar.

(b) It is clear from part (a) that

$$\lim_{x \to a^-} f(x) \le f(a) \le \lim_{x \to a^+} f(x).$$

If $\lim_{x \to a} f(x)$ exists, it follows that

$$\lim_{x \to a} f(x) = \lim_{x \to a^-} f(x) \le f(a) \le \lim_{x \to a^+} f(x) = \lim_{x \to a} f(x),$$

so $\lim_{x \to a} f(x) = f(a)$. Thus f is continuous at a, so f cannot have a removable discontinuity at a.

(c) If f is not continuous at some point a, then

$$\sup\{f(x) : x < a\} = \lim_{x \to a^-} f(x) < \lim_{x \to a^+} f(x) = \inf\{f(x) : x > a\}.$$

It follows that f(x) cannot have any value between $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$, except f(a), so f cannot satisfy the Intermediate Value Theorem.

9. (a) is obvious for $|||\ \ |||$, since $|cf|(x) = |c| \cdot |f(x)|$ for all x in [0, 1].

(b) We have $|f + g|(x) \le |f|(x) + |g|(x)$ for all x in [0, 1]. Since $|||f + g|||$ is $\sup\{|f + g|(x) : x \text{ in } [0, 1]\}$, there is some $x_0$ in [0, 1] with

$$|||f + g||| - |f + g|(x_0) < \epsilon,$$

which implies that

$$|||f + g||| - [|f|(x_0) + |g|(x_0)] < \epsilon.$$

Since $|f|(x_0) \leq |||f|||$ and $|g|(x_0) \leq |||g|||$, it follows that

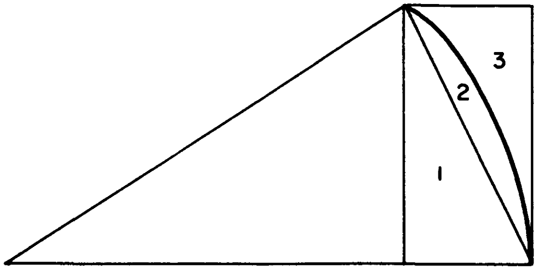
$$|||f + g||| - [|||f||| + |||g|||] < \epsilon.$$

Since this is true for every $\epsilon > 0$, it follows that $|||f + g||| \leq |||f||| + |||g|||$.

(c) follows from (b), just as in Problem 7-14.

11. (a) We have $a_{n+1} \leq a_1/2^n < a_1/n$. Choose n so that $1/n < \epsilon/a_1$. Then $a_{n+1} < \epsilon$.

(b) Let $R_i$ be the area of region number i in the following figure.



We must show that

$$R_2 < \frac{1}{2}(R_1 + R_2),$$

or

$$R_2 < R_1.$$

This is clear, since $R_2 < R_2 + R_3 = R_1$.

(c) Apply part (a) with $a_n$ = area of the circle minus the area of an inscribed regular polygon with $2^{n+1}$ sides; part (b) says that $a_{n+1} \leq a_n/2$.

(d) Let $r_1$ and $r_2$ be the radii of the two circles $C_1$ and $C_2$, and let $A_i$ be the area of the region bounded by $C_i$. We know that there are numbers $\delta_1, \delta_2 > 0$ such that

$$\left|\frac{A_1}{A_2} - \frac{B_1}{B_2}\right| < \epsilon$$

for any numbers $B_1$, $B_2$ with $|A_i - B_i| < \delta_i$. By part (c) there are numbers $n_i$ such that the area of a regular polygon, with $n_i$ sides, inscribed in $C_i$ differs from $A_i$ by less than $\delta_i$. Let $P_i$ be the area of a regular polygon inscribed in $C_i$ with $\max(n_1, n_2)$ sides. Then

$$\left|\frac{A_1}{A_2} - \frac{P_1}{P_2}\right| < \epsilon,$$

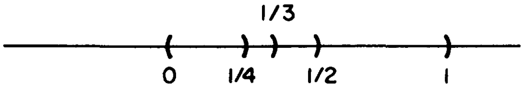
so

$$\left|\frac{A_1}{A_2} - \frac{{r_1}^2}{{r_2}^2}\right| < \epsilon.$$

Since this is true for each $\epsilon > 0$, it follows that $A_1/A_2 = r_1^2/r_2^2$.

14. (a) For each n and m we have $a_n \le b_m$, because $a_n \le a_{n+m} \le b_{n+m} \le b_m$. It follows from Problem 13 that $\sup\{a_n: n \text{ in } \mathbf{N}\} \le \inf\{b_n: n \text{ in } \mathbf{N}\}$. Let x be any number between these two numbers. Then $a_n \le x \le b_n$ for all n, so x is in every $I_n$.

(b) Let $I_n = (0, 1/n)$.



15. Let c be in each $I_n$. If $f(c) < 0$, then there is some $\delta > 0$ such that $f(x) < 0$ for all x in $[a, b]$ with $|x - c| < \delta$. Choose n with $1/2^n < \delta$. Since c is in $I_n$, which has total length $1/2^n$, it follows that all points x of $I_n$ satisfy $|x - c| < \delta$. This contradicts the fact that f changes sign on $I_n$. Similarly, we cannot have $f(c) > 0$. So $f(c) = 0$.

16. Let c be in each $I_n$. Since f is continuous at c, there is a $\delta > 0$ such that f is bounded on the set of all points in $[0, 1]$ satisfying $|x - c| < \delta$. Choose n with $1/2^n < \delta$. Since c is in $I_n$, all points x of $I_n$ satisfy $|x - c| < \delta$. This contradicts the fact that f is not bounded on $I_n$.

17. (a) (i) If $x$ is in $A$ then $x < \alpha$. So $y < x < \alpha$, so $y < \alpha$, so $y$ is in $A$.

(ii) $\alpha - 1$ is in $A$.

(iii) $\alpha + 1$ is not in $A$.

(iv) If $x$ is in $A$, then $x < \alpha$. Let $x' = (x + \alpha)/2$. Then $x < x' < \alpha$, so $x'$ is in $A$.

(b) According to (iii) there is some $y$ with $y$ not in $A$. If $y < x$, then $x$ cannot be in $A$, because (i) would imply that $y$ is in $A$. Thus $y$ is an upper bound for $A$, and $A \neq \emptyset$, by (ii), so sup $A$ exists. Given $x$ in $A$, choose $x'$ in $A$ with $x < x'$, by (iv). Then $x < x' \le \sup A$, so $x < \sup A$. Conversely, if $x < \sup A$, then there is some $y$ in $A$ with $x < y$. Hence $x$ is in $A$, by (i).

18. (a)

| | Almost upper bounds | Almost lower bounds |
|---|---|---|
| (i) | All $\alpha > 0$. | All $\alpha \le 0$. |
| (ii) | All $\alpha > 0$. | All $\alpha < 0$. |
| (iii) | All $\alpha > 0$. | All $\alpha \le 0$. |
| (iv) | All $\alpha \ge \sqrt{2}$. | All $\alpha \le 0$. |
| (v) | None. | None. |
| (vi) | All $\alpha \ge [-1 + \sqrt{5}]/2$. | All $\alpha \le [-1 + \sqrt{5}]/2$. |
| (vii) | All $\alpha \ge 0$. | All $\alpha \le [-1 + \sqrt{5}]/2$. |
| (viii) | All $\alpha > 1$. | All $\alpha \le -1$. |

(b) Every upper bound for A is surely an almost upper bound, so $B \neq \emptyset$. No lower bound for A can possibly be an almost lower bound (since A is infinite), so B is bounded below by any lower bound for A.

(c) (i), (ii), (iii) 0.

(iv) $\sqrt{2}$.

(v) Does not exist.

(vi) $[-1 + \sqrt{5}]/2$.

(vii) 0.

(viii) 1.

(d) $\underline{\lim}\, A = \sup C$, where C is the set of all almost lower bounds.

(i), (ii), (iii), (iv) 0.

(v) Does not exist.

(vi) $[-1 + \sqrt{5}]/2$.

(vii) $[-1 + \sqrt{5}]/2$.

(viii) -1.

19. (a) If x is an almost lower bound of A, and y is an almost upper bound, then there are only finitely many numbers in A which are $< x$ or $> y$. Since A is infinite, it follows that we must have $x \leq y$. Thus (Problem 12) $\underline{\lim}\, A \leq \overline{\lim}\, A$.

(b) This is clear, since $\overline{\lim} A \leq \alpha$ for any almost upper bound $\alpha$, and $\alpha = \sup A$ is an almost upper bound.

(c) If $\overline{\lim} A < \sup A$, there is some almost upper bound $x$ of $A$ with $x < \sup A$. So there are only finitely many numbers of $A$ which are greater than $x$ (and there is at least one, since $x < \sup A$). The largest of these finitely many elements is the largest element of $A$.

(d) Reverse the inequalities in the arguments for parts (b) and (c).

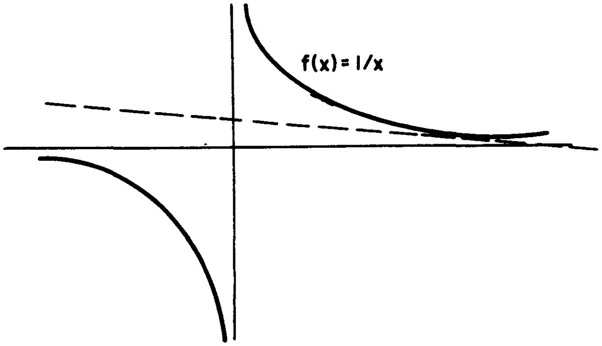
20. (a) Notice that we must have $f(x) \leq f(\sup A)$, because $f$ is continuous at $\sup A$ and there are points $y$ arbitrarily close to $\sup A$ with $f(x) \leq f(y)$. (A simple $\epsilon$-$\delta$ argument is being supressed.) Now suppose that $\sup A < b$. Then $f(b) < f(x)$. Moreover, $\sup A$ is a shadow point, so there is some $z > \sup A$ with $f(z) > f(\sup A) \geq f(x)$. We cannot have $z \leq b$, for this would mean that $z$ is in $A$. So $z > b$ and $f(b) < f(x) \leq f(z)$, contradicting the fact that $b$ is not a shadow point.

(b) Since $f$ is continuous at $a$, and $f(x) \leq f(b)$ for all $x$ in $(a, b)$, it follows that $f(a) \leq f(b)$ (either by a simple $\epsilon$-$\delta$ argument, or using Problem 6, if you prefer).

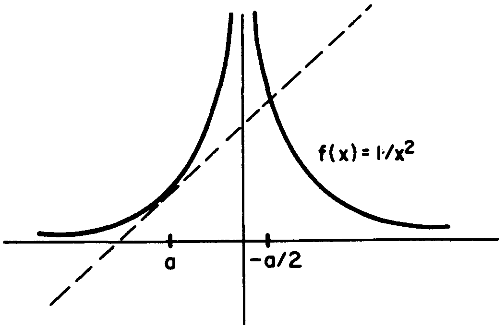
(c) If $f(a) < f(b)$, then $a$ would be a shadow point, so $f(a) = f(b)$.

# CHAPTER 9

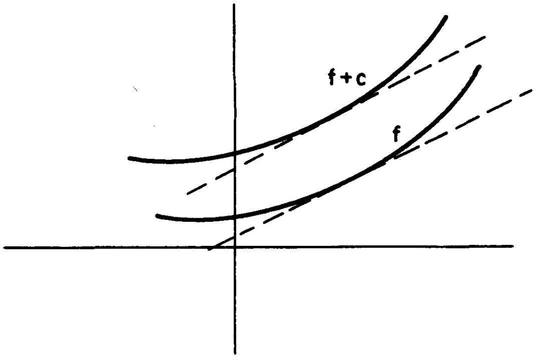
1. (b) The following figure illustrates the tangent lines to the graph of $f(x) = 1/x$.



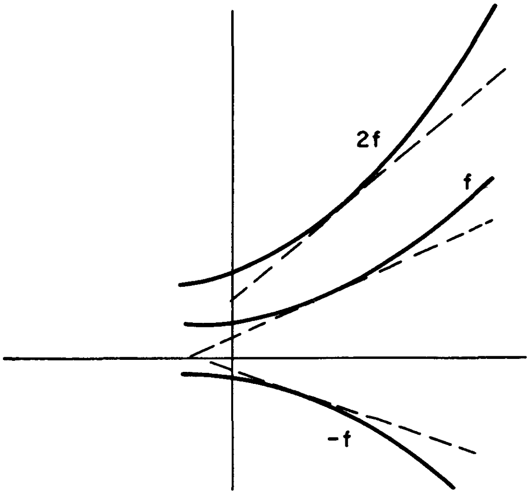
2. (b) The following figure illustrates the tangent lines to the graph of $f(x) = 1/x^2$.



6. (a) The picture below indicates the relation between $f'$ and $(f + c)'$.

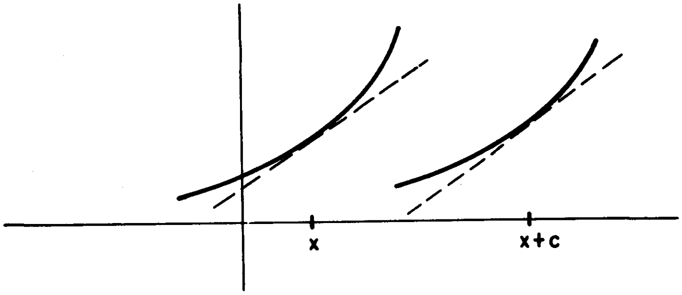


(b) The figure below indicates the relation between f' and (cf)'.



8. (a) The figure below indicates the relation between f' and g' if

$g(x) = f(x + c)$.



12. (a) $a'(t) = L(a(t))$ (the velocity at time $t$ should be the velocity allowed at the point $a(t)$, where the car is located).

(b) The hypothesis means that $b(t) = a(t - 1)$. Thus

$$b'(t) = a'(t - 1) = L(a(t - 1)) = L(b(t)).$$

(c) Suppose $b(t) = a(t) - c$. Then $b'(t) = a'(t) = L(a(t))$, whereas $b'(t)$ should be $L(b(t)) = L(a(t) - c)$. So B travels at the speed limit if the function $L$ is periodic, with period $c$.

13. The limit $\lim\limits_{t\to 0} \frac{h(a+t) - h(a)}{t}$ exists, because

$$\lim_{t\to 0^+} \frac{h(a+t) - h(a)}{t} = \lim_{t\to 0^+} \frac{g(a+t) - g(a)}{t}$$

$$= \text{right-hand derivative of } g,$$

$$\lim_{t \to 0^-} \frac{h(a+t) - h(a)}{t} = \lim_{t \to 0^-} \frac{f(a+t) - f(a)}{t}$$

$$= \text{left-hand derivative of } f,$$

and these two limits are equal.

14.

$$f'(0) = \lim_{h \to 0} \frac{f(x) - f(0)}{h} = \lim_{h \to 0} \frac{f(h)}{h}.$$

Now

$$\frac{f(h)}{h} = \begin{cases} 0, & h \text{ irrational} \\ \dfrac{h^2}{h} = h, & h \text{ rational}, \end{cases}$$

so $\lim\limits_{h \to 0} f(h)/h = 0$.

15. (a) Notice that $f(0) = 0$. Since $|f(h)/h| \leq h^2/|h| \leq |h|$, it follows that $\lim\limits_{h \to 0} f(h)/h = 0$, i.e., $f'(0) = 0$.

(b) If $g(0) = 0$ and $g'(0) = 0$, then $f'(0) = 0$: For $|f(h)/h| \leq |g(h)/h| = |[g(h) - g(0)]/h|$, which can be made as small as desired, by choosing $h$ sufficiently small, since $g'(0) = 0$.

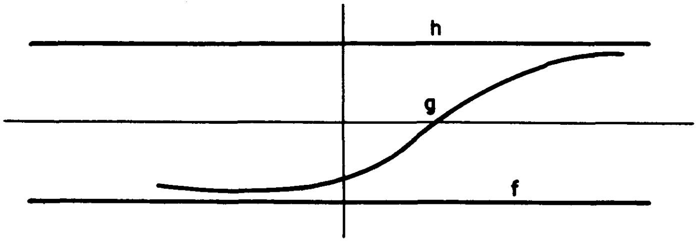
16. Since $|f(0)| \leq |0|^\alpha$, we have $f(0) = 0$. Now $|f(h)/h| \leq |h|^{\alpha-1}$, and $\lim\limits_{h \to 0} |h|^{\alpha-1} = 0$, since $\alpha > 1$, so $\lim\limits_{h \to 0} f(h)/h = 0$. Thus $f'(0) = 0$.

17. $|f(h)/h| \geq |h|^{\beta-1}$; since $\beta - 1 < 0$, the number $|h|^{\beta-1}$ becomes large as h approaches 0, so $\lim_{h \to 0} f(h)/h$ does not exist.

18. Since f is not continuous at a if a is rational, f is also not differentiable at rational a. If $a = 0.a_1a_2a_3\ldots$ is irrational and h is rational, then $a + h$ is irrational, so $f(a + h) - f(a) = 0$. But if $h = -0.00\ldots0\, a_{n+1}a_{n+2}\ldots$, then $a + h = 0.a_1a_2\ldots a_n000\ldots$, so $f(a + h) \geq 10^{-n}$, while $|h| < 10^{-n}$, so $|[f(a + h) - f(a)]/h| \geq 1$. Thus $[f(a + h) - f(a)]/h$ is 0 for arbitrarily small h and also has absolute value $\geq 1$ for arbitrarily small h, so $\lim_{h \to 0} [f(a + h) - f(a)]/h$ cannot exist.

19. (a) $\dfrac{f(a + t) - f(a)}{t} \leq \dfrac{g(a + t) - g(a)}{t} \leq \dfrac{h(a + t) - h(a)}{t}$, since $f(a) = g(a) = h(a)$. The left and right sides approach the same limit, so the middle term must also approach the same limit.

(b) A counterexample without the condition $f(a) = g(a) = h(a)$ is shown below.



20. (a) $d(x) = f(x) - f(a)(x - a) - f(a) = x^4 - 4a^3(x - a) - a^4$

$= x^4 - 4a^3x + 3a^4 = (x - a)(x^3 + ax^2 + a^2x - 3a^3)$

$= (x - a)(x - a)(x^2 + 2ax + 3a^2)$.

(b) $f(x) - f(a)$ clearly has $a$ as a root, so $f(x) - f(a)$ is divisible by $x - a$ by Problem 3-7. This means that $[f(x) - f(a)]/(x - a)$ is a polynomial function, so $d(x)/(x - a)$ is the polynomial function $h(x) = [f(x) - f(a)]/(x - a) - f'(a)$. Then $\lim_{x \to a} h(x) = 0$, by the definition of $f'(a)$. This implies that $h(a) = 0$, since the (polynomial) function $h$ is continuous. So $d(x)/(x - a)$ has $a$ as a root, so $d(x)/(x - a)$ is divisible by $(x - a)$, i.e., $d(x)$ is divisible by $(x - a)^2$.

22. (a)

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{f(x - h) - f(x)}{-h} = \lim_{h \to 0} \frac{f(x) - f(x - h)}{h}.$$

So

$$\lim_{h \to 0} \frac{f(x + h) - f(x - h)}{2h} = \frac{1}{2}\left[\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} + \lim_{h \to 0} \frac{f(x) - f(x - h)}{h}\right]$$

$$= f'(x).$$

(b)

$$\frac{f(x + h) - f(x - k)}{h + k} = \frac{h}{h + k} \cdot \frac{f(x + h) - f(x)}{h} + \frac{k}{h + k} \cdot \frac{f(x) - f(x - k)}{k}.$$

Since $[f(x + h) - f(x)]/h$ and $[f(x) - f(x - k)]/k$ are close to $f'(x)$ when h and k are sufficiently small, this would seem to imply that

$$\frac{f(x + h) - f(x - k)}{h + k} \quad \text{is close to} \quad \left(\frac{h}{h + k} + \frac{k}{h + k}\right)f'(x) = f'(x).$$

However, some care is required to carry this argument out, for the following reason. If $h/(h + k)$ were very large, then

$$\frac{h}{h + k} \cdot \frac{f(x + h) - f(x)}{h}$$

could differ from $hf'(x)/(h + k)$ by a large amount, even if $[f(x + h) - f(x)]/h$ differed from $f'(x)$ by only a small amount. It will be essential to use the fact that both h and k are positive; otherwise $h/(h + k)$ could be made very large by choosing k close to $-h$. In fact, the theorem is false if h and k are allowed to have different signs, even when $h + k = 0$ is not allowed. The proper argument is as follows. If $\epsilon > 0$ there is $\delta > 0$ such that for $0 < h < \delta$ and $0 < k < \delta$ we have

$$-\epsilon < \frac{f(x + h) - f(x)}{h} - f'(x) < \epsilon,$$

$$-\epsilon < \frac{f(x) - f(x - k)}{k} - f'(x) < \epsilon.$$

Since $h,k > 0$, we can multiply these inequalities by $h/(h + k)$ and $k/(h + k)$, respectively. Upon adding we obtain

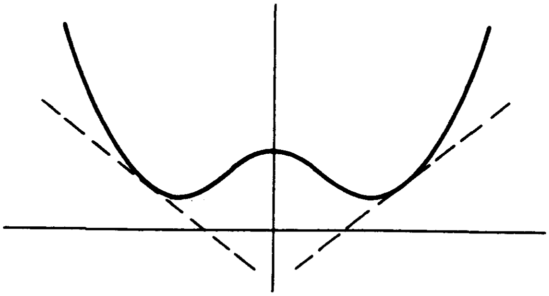
$$-\epsilon\left(\frac{h}{h+k}+\frac{k}{h+k}\right) < \frac{f(x+h)-f(x-k)}{h+k} - \left(\frac{h}{h+k}+\frac{k}{h+k}\right)f'(x) < \epsilon\left(\frac{h}{h+k}+\frac{k}{h+k}\right),$$

or

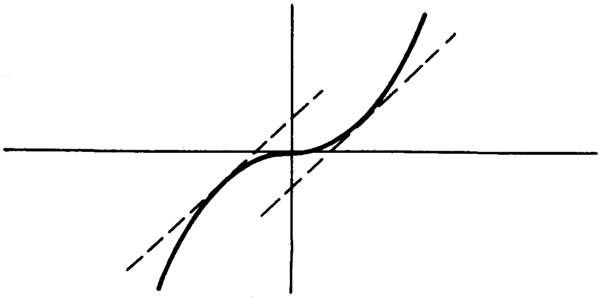
$$-\epsilon < \frac{f(x+h)-f(x-k)}{h+k} - f'(x) < \epsilon.$$

This proves the required limit.

23. If $g(x) = f(-x)$, then $g'(x) = -f'(-x)$, by Problem 8(b). But also $g(x) = f(x)$, so $g'(x) = f'(x)$, so $f'(x) = -f'(-x)$.



24. If $g(x) = f(-x)$, then $g'(x) = -f'(-x)$. But also $g(x) = -f(x)$, so $g'(x) = -f'(x)$, so $f'(x) = f'(-x)$.



25. $f^{(k)}$ is even if $k$ is even and $f$ is even, or if $k$ is odd and $f$ is odd; $f^{(k)}$ is odd in the other two cases.

26. (ii) $f''(x) = 20x^4$.

(iv) $f''(x) = 20(x - 3)^3$.

27. Proof by induction on k. The result is true for $k = 0$. If

$$S_n^{(k)}(x) = \frac{n!}{(n - k)!} x^{n-k},$$

then

$$S_n^{(k+1)}(x) = \frac{n!(n - k)}{(n - k)!} x^{n-k-1}$$

$$= \frac{n!}{[n - (k + 1)]!} x^{n-(k+1)}.$$

28. (a) Since

$$f(x) = \begin{cases} x^3, & x > 0 \\ -x^3, & x < 0, \end{cases}$$

we have

$$f'(x) = \begin{cases} 3x^2, & x > 0 \\ -3x^2, & x < 0 \end{cases} \qquad f''(x) = \begin{cases} 6x, & x > 0 \\ -6x, & x < 0. \end{cases}$$

Moreover, $f'(0) = f''(0) = 0$. But $f'''(0)$ does not exist.

(b) The same sort of reasoning shows that

$$f'(x) = \begin{cases} 4x^3, & x > 0 \\ -4x^3, & x < 0 \end{cases} \qquad f''(x) = \begin{cases} 12x^2, & x > 0 \\ -12x^2, & x < 0 \end{cases} \qquad f'''(x) = \begin{cases} 24x, & x > 0 \\ -24x, & x < 0 \end{cases}$$

and that $f'(0) = f''(0) = f'''(0) = 0$, but that $f^{(4)}(0)$ does not exist.

29. Clearly $f^{(k)}(x) = n!/(n-k)!x^{n-k}$ for $0 \le k \le n-1$ and $x > 0$, while $f^{(k)}(x) = 0$ for all $k$ if $x < 0$. From these formulas it is easy to see that $f^{(k)}(0) = 0$ for $0 \le k \le n-1$. In particular, $f^{(n-1)}(x) = n!x$ for $x \ge 0$, and $f^{(n-1)}(x) = 0$ for $x \le 0$. So $f^{(n)}(0)$ does not exist, since $\lim_{h\to 0^+} n!h/h = n!$, while $\lim_{h\to 0^-} 0/h = 0$.

30. (ii) means that $f'(a) = -1/a^2$ if $f(x) = 1/x$.

(iv) means that $g'(a) = cf'(a)$ if $g(x) = cf(x)$.

(vi) means that $f'(a^2) = 3a^4$ if $f(x) = x^3$.

(viii) means that $g'(b) = cf'(cb)$ if $g(x) = f(cx)$.

(x) means that $f^{(k)}(a) = k!\binom{n}{k}a^{n-k}$ if $f(x) = x^n$.

# CHAPTER 10

1. (ii) $\cos x + 2x \cos x^2.$

(iv) $\cos(\sin x)\cdot\cos x.$

(vi) $\dfrac{x \cos(\cos x)(-\sin x) - \sin(\cos x)}{x^2}.$

(viii) $\cos(\cos(\sin x))\cdot(-\sin(\sin x))\cdot\cos x.$

2. (ii) $3 \sin^2(x^2 + \sin x)\cdot\cos(x^2 + \sin x)\cdot(2x + \cos x).$

(iv) $\cos\left(\dfrac{x^3}{\cos x^3}\right)\cdot \dfrac{(\cos x^3)\ 3x^2 + x^3 \sin x^3 \cdot 3x^2}{\cos^2 x^3}.$

(vi) $31^2(\cos x)^{31^2-1}\cdot(-\sin x).$

(viii) $3 \sin^2(\sin^2 (\sin x))\cdot\cos(\sin^2(\sin x))\cdot 2 \sin(\sin x)\cdot\cos(\sin x)\cdot\cos x.$

(x) $\cos(\sin(\sin(\sin(\sin x))))\cdot\cos(\sin(\sin(\sin x)))\cdot\cos(\sin(\sin x))$
$\cdot\cos(\sin x)\cdot\cos x.$

(xii) $5((x^2 + x)^3 + x)^4\cdot[1 + 4((x^2 + x)^3 + x)^3 \{1 + 3(x^2 + x)^2[1 + 2x]\}].$

(xiv) $\cos(6 \cos(6 \sin(6 \cos 6x)))\cdot 6(-\sin(6 \sin(6 \cos 6x))\cdot 6 \cos(6 \cos 6x)$
$\cdot 6(-\sin 6x)\cdot 6.$

(xvi) $$-\frac{\left[1 - \frac{-2(1 + \cos x)}{(x + \sin x)^2}\right]}{\left[x - \frac{2}{x + \sin x}\right]^2}.$$

(xviii) $$\cos\left(\frac{x}{x - \sin\left(\frac{x}{x - \sin x}\right)}\right) \cdot \frac{x - \sin\left(\frac{x}{x - \sin x}\right) - x\left[1 - \cos\left(\frac{x}{x - \sin x}\right)\frac{x - \sin x - x[1 - \cos x]}{(x - \sin x)^2}\right]}{\left[x - \sin\left(\frac{x}{x - \sin x}\right)\right]^2}.$$

3. See page 263 of the text.

4. (ii) $\cos(\sin x)$.

   (iv) $0$.

5. (ii) $(2x)^2$.

   (iv) $17 \cdot 17$.

6. (ii) $f'(x) = g'(x \cdot g(a)) \cdot g(a)$.

   (iv) $f'(x) = g'(x)(x - a) + g(x)$.

   (vi) $f'(x) = g'((x - 3)^2) \cdot 2(x - 3)$.

8. (ii) $(k \circ f)'(0) = k'(f(0)) \cdot f'(0) = 0.$

9. By definition,

$$f'(0) = \lim_{x \to 0} \frac{f(x) - 0}{x}$$

$$= \lim_{x \to 0} \frac{g(x) \sin 1/x}{x}.$$

Now

$$\lim_{x \to 0} \frac{g(x)}{x} = \lim_{x \to 0} \frac{g(x) - g(0)}{x}$$

$$= g'(0) = 0.$$

Since $|\sin 1/x| \le 1$, it follows that $f'(0) = 0$ (as in Problem 5-19).

11. (a) The Chain Rule and Problem 9-3 imply that

$$f'(x) = \frac{1}{2\sqrt{1 - x^2}} \cdot -2x$$

$$= -\frac{x}{\sqrt{1 - x^2}}.$$

(b) The tangent line through $(a, \sqrt{1 - a^2})$ is the graph of

$$g(x) = -\frac{a}{\sqrt{1 - a^2}}(x - a) + \sqrt{1 - a^2}.$$

So if $f(x) = g(x)$, then

$$\sqrt{1 - x^2} = \frac{-a}{\sqrt{1 - a^2}}(x - a) + \sqrt{1 - a^2}.$$

Squaring yields

$$1 - x^2 = \frac{a^2(x - a)^2}{1 - a^2} - 2a(x - a) + 1 - a^2.$$

Multiplying through by $1 - a^2$, and multiplying out, everything reduces to

$$-x^2 - a^2 = -2ax,$$

i.e., $(x - a)^2 = 0$, so $x = a$. Notice that the same argument shows that $g$ does not intersect the graph of $f(x) = -\sqrt{1 - x^2}$, which is the bottom half of the unit circle.

12. The graph of the function

$$f(x) = b\sqrt{1 - \frac{x^2}{a^2}}$$

is the top half of the ellipse which consists of all points $(x, y)$ satisfying

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Now

$$f'(x) = \frac{-bx}{a^2\sqrt{1 - \frac{x^2}{a^2}}}.$$

If the tangent line through $(c, b\sqrt{1 - c^2/a^2})$ intersects the graph of f at x, then

$$b\sqrt{1 - \frac{x^2}{a^2}} = \frac{-bc}{a^2\sqrt{1 - \frac{c^2}{a^2}}}(x - c) + b\sqrt{1 - \frac{c^2}{a^2}}.$$

The easiest way to solve this equation is to use the following trick. If we let $x' = x/a$ and $c' = c/a$, then the equation becomes

$$(1) \qquad b\sqrt{1 - (x')^2} = \frac{-b(c'a)}{a^2\sqrt{1 - (c')^2}}(x' - c')\cdot a + b\sqrt{1 - (c')^2},$$

or simply

$$\sqrt{1 - (x')^2} = \frac{-c'}{\sqrt{1 - (c')^2}}(x' - c') + \sqrt{1 - (c')^2}.$$

The solution to Problem 11 shows that $x' = c'$, so $x = c$.

For the hyperbola, we consider

$$f(x) = b\sqrt{\frac{x^2}{a^2} - 1}.$$

Then

$$f'(x) = \frac{bx}{a^2\sqrt{\frac{x^2}{a^2} - 1}},$$

so if the tangent line through $(c, b\sqrt{c^2/a^2 - 1})$ intersects the graph at x, then

(2) $$b\sqrt{\frac{x^2}{a^2}-1} = \frac{bc}{a^2\sqrt{\frac{c^2}{a^2}-1}}(x-c) + b\sqrt{\frac{c^2}{a^2}-1}.$$

Squaring equations (1) and (2) produces the same result, so the solutions of (2) are also $x = c$.

13. No. For example, $g$ might be $-f$.

If $f(a) \neq 0$ and $f \cdot g$ and $f$ are differentiable at $a$, then $g$ is differentiable at $a$.

14. (a) Since $f$ is differentiable at $a$, it is continuous at $a$. Since $f(a) \neq 0$, it follows that $f(x) \neq 0$ for all $x$ in an interval around $a$. So $f = |f|$ or $f = -|f|$ in this interval, so $|f|'(a) = f'(a)$ or $|f|'(a) = -f'(a)$. (It is also possible to use the Chain Rule, and Problem 9-3: $|f| = \sqrt{f^2}$, so

$$\begin{aligned}|f|'(x) &= \frac{1}{2\sqrt{f(x)^2}} \cdot 2f(x)f'(x)\\ &= f'(x) \cdot \frac{f(x)}{|f(x)|}.)\end{aligned}$$

(b) Let $f(x) = x - a$.

(c) This follows from part (a), since $\max(f, g) = [f + g + |f - g|]/2$ and $\min(f, g) = [f + g - |f - g|]/2$.

(d) Use the same example as in part (b), choosing $g = 0$.

15. The proof is by induction on n. For $n = 1$, Leibnitz's formula is Theorem 4. Suppose that for a certain n, Leibnitz's formula is true for all numbers a such that $f^{(n)}(a)$ and $g^{(n)}(a)$ exist. Suppose that $f^{(n+1)}(a)$ and $g^{(n+1)}(a)$ exist. Then $f^{(n)}(x)$ and $g^{(n)}(x)$ must exist for all x in some interval around a. So Leibnitz's formula holds for all these x, that is,

$$(f\cdot g)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x)\cdot g^{(n-k)}(x)$$

for all x in some interval around a. Differentiating, and using Theorem 4, we find that

$$\begin{aligned}(f\cdot g)^{(n+1)}(a) &= \sum_{k=0}^{n} \binom{n}{k} (f^{(k)} \cdot g^{(n-k)})'(a)\\ &= \sum_{k=0}^{n} \binom{n}{k} [f^{(k+1)}(a)g^{(n-k)}(a) + f^{(k)}(a)g^{(n+1-k)}(a)]\\ &= \sum_{k=1}^{n+1} \binom{n}{k-1} f^{(k)}(a)g^{(n+1-k)}(a) + \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(a)g^{(n+1-k)}(a)\\ &= \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(k)}(a)g^{(n+1-k)}(a) \qquad \text{by Problem 2-3(a).}\end{aligned}$$

16. The formulas

$$(f\circ g)'(x) = f'(g(x))\cdot g'(x),$$

$$(f\circ g)''(x) = f''(g(x))\cdot [g'(x)]^2 + f'(g(x))\cdot g''(x),$$

$$(f\circ g)'''(x) = f'''(g(x))\cdot [g'(x)]^3 + 2f''(g(x))\cdot g'(x)\cdot g''(x) + f''(g(x))\cdot g'(x)\cdot g''(x) + f'(g(x))\cdot g'''(x),$$

lead to the following conjecture: If $f^{(n)}(g(a))$ and $g^{(n)}(a)$ exist, then $(f\circ g)^{(n)}(a)$ exists and is a sum of terms of the form

$$c\cdot [g'(a)]^{m_1}\cdot \ldots \cdot [g^{(n)}(a)]^{m_n} \cdot f^{(k)}(g(a)),$$

for some number c, non-negative integers $m_1,\ldots,m_n$, and natural number $k \le n$. To prove this assertion by induction, note that it is true for $n = 1$ (with $a = m_1 = k = 1$). Now suppose that for a certain n, this assertion is true for all numbers a such that $f^{(n)}(g(a))$ and $g^{(n)}(a)$ exist. Suppose that $f^{(n+1)}(g(a))$ and $g^{(n+1)}(a)$ exist. Then $g^{(k)}(x)$ must exist for all $k \le n$ and all x in some interval around a, and $f^{(k)}(y)$ must exist for all $k \le n$ and all y in some interval around $g(a)$. Since g is continuous at a, this implies that $f^{(k)}(g(x))$ exists for all x in some interval around a. So the assertion is true for all these x, that is, $(f\circ g)^{(n)}(x)$ is a sum of terms of the form

$$c\cdot [g'(x)]^{m_1}\cdot \ldots \cdot [g^{(n)}(x)]^{m_n}\cdot f^{(k)}(g(a)), \qquad m_1,\ldots,m_n \ge 0,\ 1 \le k \le n.$$

Consequently, $(f\circ g)^{(n+1)}(a)$ is a sum of terms of the form

$$c\cdot m_\alpha[g'(a)]^{m_1}\cdot\ldots\cdot[g^{(\alpha)}(a)]^{m_\alpha-1}\cdot\ldots\cdot[g^{(n)}(a)]^{m_n}\cdot f^{(k)}(g(a)) \qquad m_\alpha > 0$$

or

$$c\cdot[g'(a)]^{m_1+1}\cdot\ldots\cdot[g^{(n)}(a)]^{m_n}\cdot f^{(k+1)}(g(a)).$$

17. (a) We can choose

$$g(x) = \frac{a_n x^{n+1}}{n+1} + \frac{a_{n-1}x^n}{n} + \ldots + \frac{a_1 x^2}{2} + a_0 x + c$$

for any number c.

(b) Let

$$g(x) = \frac{b_2 x^{-1}}{-1} + \frac{b_3 x^{-2}}{-2} + \ldots + \frac{b_m x^{-m+1}}{-m+1}.$$

(c) No, the derivative of f is

$$f'(x) = na_n x^{n-1} + \ldots + a_1 - \frac{b_1}{x^2} - \frac{2b_2}{x^3} - \ldots - \frac{mb_m}{x^{m+1}}.$$

18. (a) Let g be a polynomial function of degree n - 1 with precisely n - 1 roots (as in Problem 4-7(d)); then g = f' for some polynomial function f of degree n (Problem 17).

(b) Proceed as in part (a), starting with a polynomial function g of degree n - 1 with no roots (notice that n - 1 is even).

(c) We can proceed as in part (a), or simply note that $f(x) = x^n$ has the desired property.

(d) Proceed as in part (a), starting with a polynomial function g of degree $n - 1$ with k roots (this exists by Problem 7-4).

19. (a) If a is a double root of f, so that $f(x) = (x - a)^2 g(x)$, then $f'(x) = (x - a)^2 g'(x) + 2(x - a)g(x)$, so $f'(a) = 0$. Conversely, if $f(a) = 0$ and $f'(a) = 0$, then $f(x) = (x - a)g(x)$ for some g, and $f'(x) = (x - a)g'(x) + g(x)$, so $0 = f'(a) = g(a)$; thus $g(x) = (x - a)h(x)$, so $f(x) = (x - a)^2 h(x)$.

(b) The only root of $0 = f'(x) = 2ax + b$ is $x = -b/2a$, so f has a double root if and only if

$$\begin{aligned} 0 = f(-b/2a) &= a\left(\frac{b^2}{4a^2}\right) + b\left(\frac{-b}{2a}\right) + c \\ &= -\frac{b^2}{4a} + c, \end{aligned}$$

or $b^2 - 4ac = 0$. Geometrically, this is precisely the condition that the graph of f touches the horizontal axis at the single point $-b/2a$ (compare with Figure 22 in Problem 9-20).

20. Since $d'(x) = f'(x) - f'(a)$, we have $d'(a) = 0$. So $a$ is a double root of $d$.

21. (a) Clearly $f$ will have to be of the form

$$f(x) = \prod_{\substack{j=1 \\ j \neq i}}^{n} (x - x_j)^2 (ax + b)$$

(because each $x_j$, $j \neq i$ is a double root, by Problem 19). It therefore suffices to show that $a$ and $b$ can be picked so that $f(x_i) = a_i$ and $f'(x_i) = b_i$. If we write $f$ in the form $f(x) = g(x)(ax + b)$, then we must solve

$$[g(x_i)x_i]\cdot a + g(x_i) \cdot b = a_i$$

$$[g'(x_i)x_i + g(x_i)]\cdot a + g'(x_i)\cdot b = b_i.$$

These equations can always be solved because

$$[g(x_i)x_i]\cdot g'(x_i) - [g'(x_i)x_i + g(x_i)]g(x_i) = [g(x_i)]^2 \neq 0.$$

(b) Let $f_i$ be the function constructed in part (a), and let $f = f_1 + \dots + f_n$.

22. (a) If $g(a)$ and $g(b)$ had different signs, then $g(x)$ would be 0 for some $x$ in $(a, b)$, which implies that $f(x) = 0$, contradicting the fact that $a$ and $b$ are consecutive roots.

(b) We have

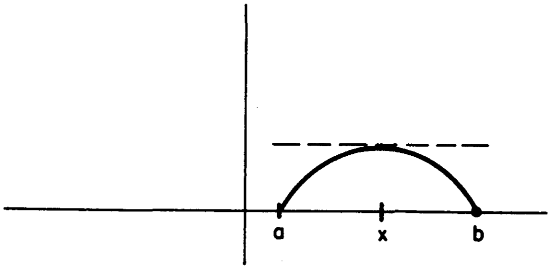
$$f'(x) = (x - b)g(x) + (x - a)g(x) + (x - a)(x - b)g'(x),$$

so

$$f'(a) = (a - b)g(a),$$

$$f'(b) = (b - a)g(b).$$

Since $g(a)$ and $g(b)$ have the same sign, $f'(a)$ and $f'(b)$ have different signs. So $f'(x) = 0$ for some $x$ in $(a, b)$, since $f'$ is a continuous function.



(c) Since

$$f'(x) = m(x - a)^{m-1} (x - b)^n g(x) + (x - a)^m n(x - b)^{n-1} g(x)$$
$$+ (x - a)^m(x - b)^n g'(x),$$

we have

$$h(a) = m(a - b)g(a),$$

$$h(b) = n(a - b)g(b),$$

so $h(a)$ and $h(b)$ have different signs, so $h(x) = 0$ for some $x$ in $(a, b)$, which implies that $f'(x) = 0$.

23.

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(h) - f(0)}{h} \\ &= \lim_{h \to 0} \frac{hg(h) - 0}{h} \\ &= \lim_{h \to 0} g(h) = g(0) \qquad \text{since } g \text{ is continuous at } 0. \end{aligned}$$

24. Let

$$g(x) = \begin{cases} \dfrac{f(x)}{x}, & x \neq 0 \\ f'(0), & x = 0. \end{cases}$$

Then $f(x) = xg(x)$ for all $x$, and

$$g(0) = f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} g(x),$$

so $g$ is continuous at 0.

25. The proof is by induction on $k$. For $k = 1$ we have

$$\begin{aligned} f'(x) &= -nx^{-n-1} \\ &= (-1)^1 \frac{(n + 1 - 1)!}{(1 - 1)!} x^{-n-1} \qquad \text{for } x \neq 0. \end{aligned}$$

Suppose that

$$f^{(k)}(x) = (-1)^k \frac{(n + k - 1)!}{(k - 1)!} x^{-n-k} \qquad \text{for } x \neq 0.$$

Then

$$\begin{aligned} f^{(k+1)}(x) &= (-1)^k \frac{(-n - k)(n + k - 1)!}{(k - 1)!} x^{-n-k-1} \\ &= (-1)^{k+1} \frac{(n + k)!}{(k - 1)!} x^{-n-(k+1)} \qquad \text{for } x \neq 0. \end{aligned}$$

26. If $x = f(x)g(x)$, then $1 = f'(x)g(x) + f(x)g'(x)$. In particular, $1 = f'(0)g(0) + f(0)g'(0) = 0$, a contradiction.

27. (a) Using Problem 25 and the Chain Rule, we obtain

$$f^{(k)}(x) = (-1)^k \frac{(n + k - 1)!}{(k - 1)!} (x - a)^{-n-k} \qquad \text{for } x \neq a.$$

(b) Since

$$f(x) = \frac{1}{x^2 - 1} = \frac{1/2}{x - 1} - \frac{1/2}{x + 1},$$

we obtain, using part (a),

$$f^{(k)}(x) = \frac{(-1)^k(n + k - 1)!}{2(k - 1)!}[(x - 1)^{-n-k} - (x + 1)^{-n-k}].$$

28, 29. The formulas

$$\begin{aligned}
f(x) &= x^m \sin \frac{1}{x} \\
f'(x) &= mx^{m-1} \sin \frac{1}{x} - x^{m-2} \cos \frac{1}{x}, \\
f''(x) &= m(m - 1)x^{m-2} \sin \frac{1}{x} - mx^{m-3} \cos \frac{1}{x} - (m - 2)x^{m-3} \cos \frac{1}{x} \\
&\qquad - x^{m-4} \sin \frac{1}{x} \\
&= m(m - 1)x^{m-2} \sin \frac{1}{x} + (2 - 2m)x^{m-3} \cos \frac{1}{x} - x^{m-4} \sin \frac{1}{x}, \\
f'''(x) &= m(m - 1)(m - 2)x^{m-3} \sin \frac{1}{x} - m(m - 1)x^{m-4} \cos \frac{1}{x} \\
&\qquad + (m - 3)(2 - 2m)x^{m-4} \cos \frac{1}{x} + (2 - 2m)x^{m-5} \sin \frac{1}{x} \\
&\qquad - (m - 4)x^{m-5} \sin \frac{1}{x} + x^{m-6} \cos \frac{1}{x},
\end{aligned}$$

suggest the following conjecture: If $f(x) = x^m \sin 1/x$, for $x \neq 0$, then

$$f^{(k)}(x) = ax^{m-k} \sin \frac{1}{x} + \sum_{\ell=k+1}^{2k-1} (a_\ell x^{m-\ell} \sin \frac{1}{x} + b_\ell x^{m-\ell} \cos \frac{1}{x}) \pm \begin{cases} x^{m-2k} \sin \frac{1}{x}, & k \text{ even} \\ x^{m-2k} \cos \frac{1}{x}, & k \text{ odd} \end{cases}$$

for certain numbers $a$, $a_\ell$, $b_\ell$. Once this conjecture is made, it is easy to check it by induction. In fact, differentiating the first term yields

$$a(m - k)x^{m-(k+1)} \sin \frac{1}{x} - ax^{m-(k+2)} \cos \frac{1}{x},$$

and the second half of this expression can be incorporated in the sum $\sum_{\ell=k+2}^{2k+1}$ appearing in the desired expression for $f^{(k+1)}(x)$. Similarly, differentiating the last term yields

$$\begin{cases} \pm\,(m-2k)x^{m-(2k+1)}\sin\frac{1}{x} \mp x^{m-2(k+1)}\cos\frac{1}{x}, & k \text{ even} \quad (k+1 \text{ odd}) \\ \pm\,(m-2k)x^{m-(2k+1)}\cos\frac{1}{x} \pm x^{m-2(k+1)}\sin\frac{1}{x}, & k \text{ odd} \quad (k+1 \text{ even}), \end{cases}$$

and the first half of each expression can be incorporated in the sum $\sum\limits_{\ell=k+2}^{2k+1}$.

Finally, each term appearing in the sum $\sum\limits_{\ell=k+1}^{2k-1}$ yields upon differentiation two terms that can be incorporated in the new sum $\sum\limits_{\ell=k+2}^{2k+1}$.

It follows, in particular, that if $m = 2n$, then $f^{(k)}(x)$ always has a factor of at least $x^2$ for $k < n$ (while the remaining factor is bounded in an interval around 0). So if we define $f(0) = 0$, then

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(h)-f(0)}{h} \\ &= \lim_{h\to 0} \frac{f(h)}{h} = 0, \qquad \text{since } f(h) \text{ has a factor of at least } h^2; \end{aligned}$$

consequently, if $2 \le n$, then

$$\begin{aligned} f''(0) &= \lim_{h\to 0} \frac{f'(h)-f'(0)}{h} \\ &= \lim_{h\to 0} \frac{f'(h)}{h} = 0, \qquad \text{since } f'(h) \text{ has a factor of at least } h^2; \end{aligned}$$

consequently, if $3 \le n$, then $f'''(0) = 0$, etc. This argument (which is really another inductive argument) shows that $f'(0) = \dots = f^{(n)}(0) = 0$. On the other hand, $f^{(n)}(x)$ is a sum of terms which do have a factor of at least $x^2$, together with $\pm$ sin 1/x or $\pm$ cos 1/x, so $f^{(n)}$ is not continuous at 0.

If $m = 2n + 1$, then $f^{(k)}(x)$ always has a factor of at least $x^2$ for $k < n$, so $f'(0) = \ldots = f^{(n)}(0) = 0$, but $f^{(n)}(x)$ is a sum of terms which do have a factor of at least $x^2$, together with $\pm$ x cos 1/x or $\pm$ x sin 1/x. It follows that $f^{(n)}$ is continuous, but not differentiable, at 0.

30. (ii) $\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = (\cos y)\cdot(-\sin x) = \cos(\cos x)\cdot(-\sin x)$.

(iv) $\frac{dz}{dx} = \frac{dz}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx} = (\cos v)(-\sin u)(\cos x) = \cos(\cos(\sin x))$
$\cdot(-\sin(\sin x))\cdot\cos x$.

# CHAPTER 11

1. (ii) $f'(x) = 5x^4 + 1 = 0$ for no $x$;

$f(-1) = -1$, $f(1) = 3$;

maximum = 3, minimum = -1.

(iv) $f'(x) = -\dfrac{(5x^4 + 1)}{(x^5 + x + 1)^2} = 0$ for no $x$;

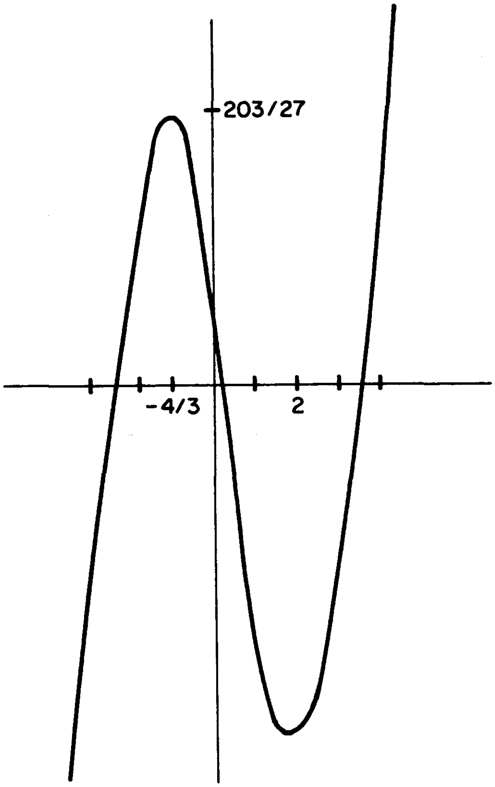
$f(-1/2) = 32/15$, $f(1) = 1/3$;

maximum = 32/15, minimum = 1/3.

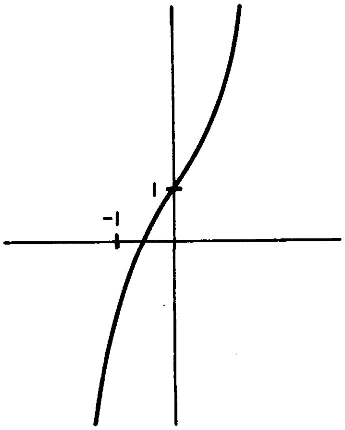
(Notice that $g(x) = x^5 + x + 1$ is increasing, since $g'(x) = 5x^4 + 1 > 0$ for all $x$; since $g(-1/2) = 15/32 > 0$, this shows that $g(x) \neq 0$ for all $x$ in $[-1/2, 1]$, so $f$ is differentiable on $[-1/2, 1]$.)

(vi) $f$ is not bounded above or below on $[0, 5]$.

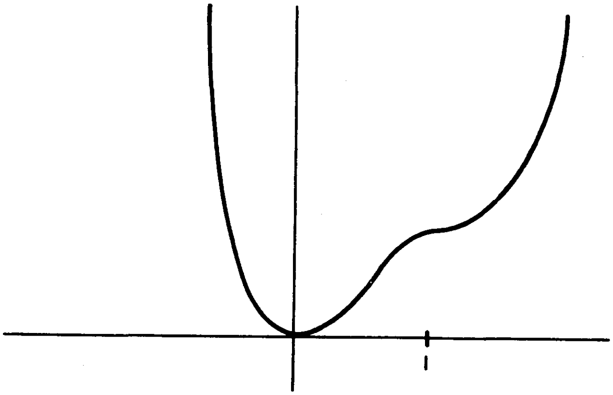
2. (i) $-4/3$ is a local maximum point, and 2 is a local minimum point.



(ii) No local maximum or minimum points.

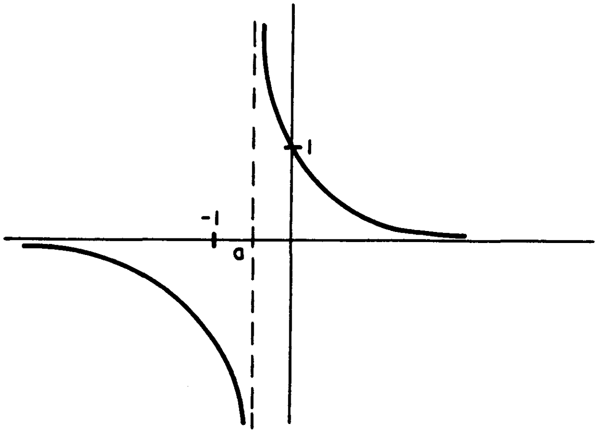


(iii) 0 is a local minimum point, and there are no local maximum points.

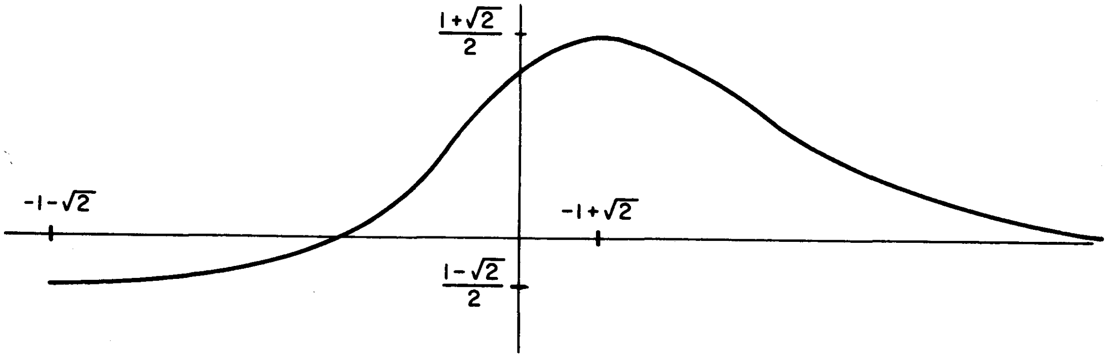


(iv) No local maximum or minimum points.

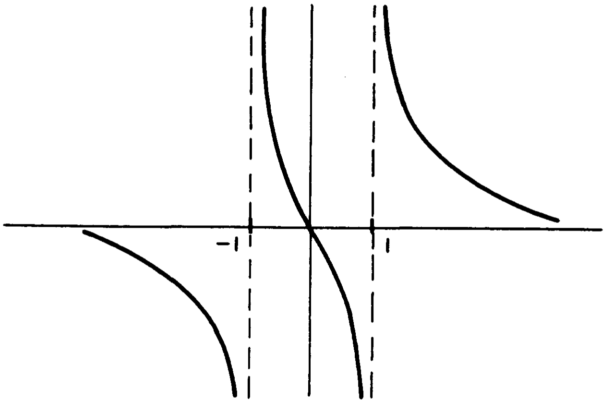
In the figure below, a is the unique root of $x^5 + x + 1 = 0$.



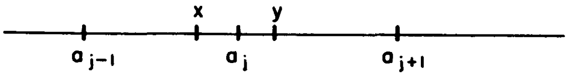
(v) $-1 + \sqrt{2}$ is a local maximum point, and $-1 - \sqrt{2}$ is a local minimum point.



(vi) No local maximum or minimum points, since $f'(x) = -(1 + x^2)/(x^2 - 1)^2 < 0$ for $x \neq \pm 1$.



3. (b) Suppose x and y are points in $[a_{j-1}, a_j]$ and $[a_j, a_{j+1}]$, respectively, with $|x - a_j| = |y - a_j|$.



Then

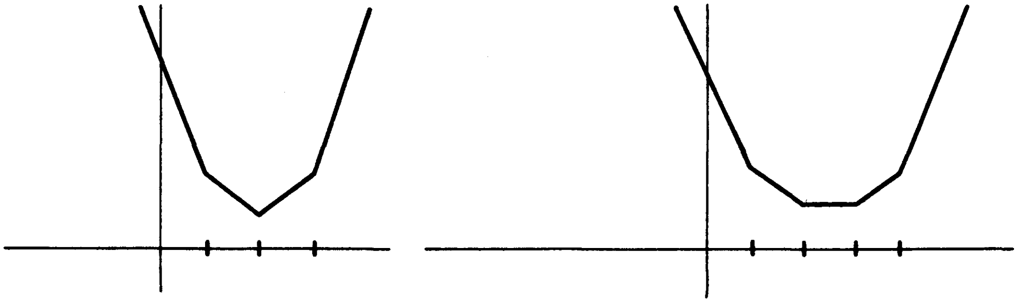
$$|y - a_i| = |x - a_i| + |y - x| \qquad \text{for} \qquad i \le j - 1,$$

$$|y - a_i| = |x - a_i| - |y - x| \qquad \text{for} \qquad i \ge j + 1.$$

So

$$\begin{aligned} f(y) &= f(x) + |y - x|\cdot\{(j - 1) - (n - j)\} \\ &= f(x) + |y - x|\cdot\{2j - n - 1\}. \end{aligned}$$

This shows that f decreases until it reaches the "middlemost $a_i$" and then increases. The minimum occurs at $a_{(n-1)/2}$ if n is odd and on the whole interval $[a_{n/2}, a_{n/2 + 1}]$ if n is even.



(c) We have

$$f(x) = \begin{cases} \dfrac{1}{1 - x} + \dfrac{1}{1 + a - x}, & x < 0 \\[2ex] \dfrac{1}{1 + x} + \dfrac{1}{1 + a - x}, & 0 < x < a \\[2ex] \dfrac{1}{1 + x} + \dfrac{1}{1 + x - a}, & a < x, \end{cases}$$

so

$$f'(x) = \begin{cases} \dfrac{1}{(1-x)^2} + \dfrac{1}{(1+a-x)^2}, & x < 0 \\ \dfrac{-1}{(1+x)^2} + \dfrac{1}{(1+a-x)^2}, & 0 < x < a \\ \dfrac{-1}{(1+x)^2} - \dfrac{1}{(1+x-a)^2}, & a < x. \end{cases}$$

Thus f is increasing on $(-\infty, 0]$ and decreasing on $[a, \infty)$, so the maximum of f on $[0, a]$ is the maximum on $\mathbf{R}$. If $f'(x) = 0$ for x in $(0, a)$, then

$$(1 + x)^2 - (1 + a - x)^2 = 0,$$

whose only solution is $x = a/2$. Since

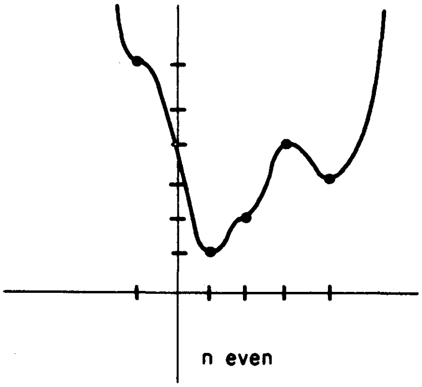
$$f\left(\frac{a}{2}\right) = \frac{4}{2 + a} < \frac{2 + a}{1 + a} = f(0) = f(a),$$

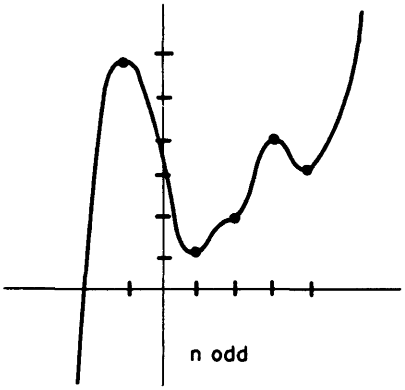
the maximum value is $(2 + a)/(1 + a)$.

4. (ii) All irrational x are local minimum points, and all rational x are local maximum points.

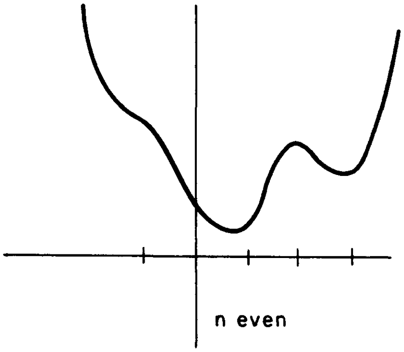
(iv) All $1/n$ for n in $\mathbf{N}$ are local maximum points, and all other x are local minimum points.

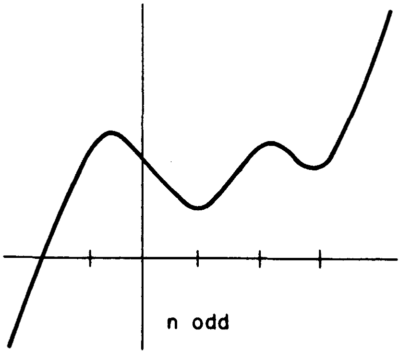
9.





10. (a) 1 is a local minimum point, and 2 is local maximum point. The nature of the critical points -1 and 3 can be determined by the behavior of f(x) for large |x|:



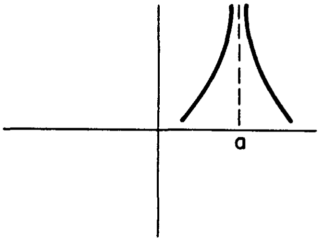


(b) No, for if 2 were the largest critical point, then f would have to be decreasing on (3, ∞), since 2 is a local maximum point.

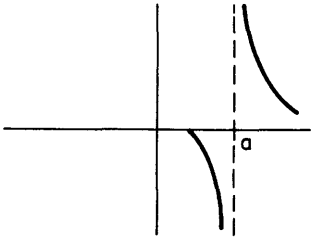
11. Let $f(x) = r(x)/s(x)$ for polynomial functions $r$ and $s$. It is possible that $r$ and $s$ have a common root $a$, but in this case $r(x) = (x - a)r_1(x)$ and $s(x) = (x - a)s_1(x)$ for certain polynomial functions $r_1$ and $s_1$ (Problem 3-7). This means that $f(a)$ is undefined but that $f(x) = r_1(x)/s_1(x)$ for $x \neq a$ (and $s_1(x) \neq 0$). After factoring out all common linear factors of $r$ and $s$, we find that the graph of $f$ consists, except for a finite number of points, of the graph of

$$g(x) = \frac{a_nx^n + a_{n-1}x^{n-1} + \dots + a_0}{b_mx^m + b_{m-1}x^{m-1} + \dots + b_0} = \frac{p(x)}{q(x)},$$

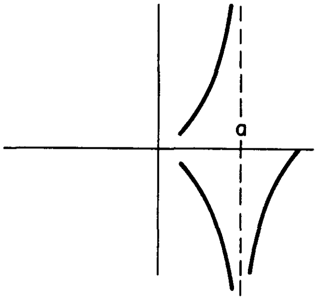
where $p$ and $q$ have no common roots. The function $g$ is defined at all points $a$ except those with $q(a) = 0$ (of which there are at most $m$). Near such a point $a$ the graph of $g$ looks like (a), (b), (c), or (d),



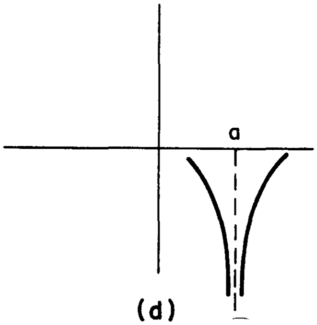
(a)



(b)



(c)



(d)

depending on the sign of $p(a)$ and whether $a$ is a local maximum or minimum point for $q$ or whether $q$ is increasing or decreasing in an interval around $a$. Since

$$g'(x) = \frac{q(x)p'(x) - p(x)q'(x)}{[q(x)]^2}$$

and $qp' - pq'$ is a polynomial function of degree at most $m + n$, there are at most $m + n$ local maximum and minimum points. On the intervals between these points and the points of discontinuity, $g$ is either increasing or decreasing. The behavior of $g(x)$ for large $x$ or large negative $x$ has been discussed in Problems 5-30 and 5-32.

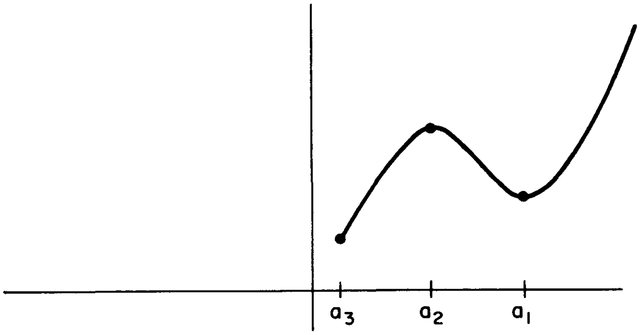
12. (a) This follows from Problem 3-7 and the fact that the difference of the two polynomial functions has degree at most $\max(m, n)$.

(b) If $m \geq n$, let $f_1$ be a polynomial function which has $m$ roots, and let $f(x) = f_1(x) + x^n$ and $g(x) = x^n$.

13. (a) The polynomial function $f'$, of degree $n - 1$, has $k$ roots, and no multiple roots, since $f''(x) \neq 0$ when $f'(x) = 0$. It follows from Problem 7-4 that $n - 1 - k$ is odd.

(b) Since $n - 1 - k$ is even, there is a polynomial function $g$ of degree $n - 1$ with exactly $k$ roots. Let $f$ be a polynomial function of degree $n$ with $f' = g$ (Problem 10-17).

(c) Let $\ell = k_1 + k_2$ and let $a_\ell < a_{\ell-1} < \dots < a_1$ be all the local maximum and minimum points. On the intervals between these points f is either decreasing or increasing. Since $\lim_{x \to \infty} f(x) = \infty$, the function f must be increasing on $(a_1, \infty)$. Thus $a_1$ must be a local minimum point. Consequently, f must be decreasing on $(a_2, a_1)$, which shows that $a_2$ must be a local maximum point.



Continuing in this way we see that $a_k$ is a local minimum point if k is odd and a local maximum point if k is even.

Now if n is even, then $a_\ell$ must be a local minimum point, since $\lim_{x \to -\infty} f(x) = \infty$. Thus $\ell$ must be odd, so $a_1, a_3, \dots, a_\ell$ are the local minimum points, and $a_2, \dots, a_{\ell-1}$ are the local maximum points. Consequently $k_2 = k_1 + 1$. If n is odd, then $a$ must be a local maximum point, since $\lim_{x \to -\infty} f(x) = -\infty$. The same sort of reasoning then shows that $k_1 = k_2$.

(d) The hypotheses imply that $n - 1 - (k_1 + k_2)$ is even. Let $\ell = [n - 1 - (k_1 + k_2)]/2$, and choose a polynomial function $f$ of degree $n$ with $f'$ as in the hint. Since $(1 + x^2)^\ell > 0$ for all $x$, it follows that $f'(x) > 0$ for $x > a_{k_1 + k_2}$ and that the sign of $f'$ changes as we go from $(a_{i-1}, a_i)$ to $(a_{i-2}, a_{i-1})$. Thus $a_{k_1 + k_2}$, $a_{k_1 + k_2 - 2}, \ldots$ are local minimum points and $a_{k_1 + k_2 - 1}$, $a_{k_1 + k_2 - 3}, \ldots$ are local maximum points.

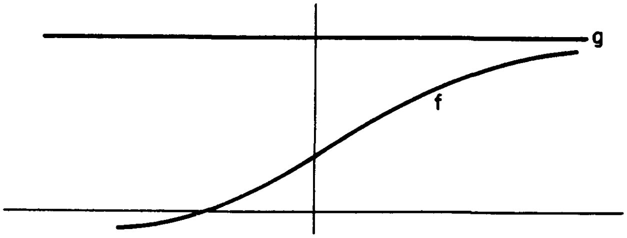
15. (a) Apply the Mean Value Theorem to $f - g$: If $x > a$, then

$$\frac{f(x) - g(x)}{x - a} = \frac{f(x) - g(x) - [f(a) - g(a)]}{x - a}$$

$$= f'(y) - g'(y) > 0 \qquad \text{for some } y \text{ in } (a, x).$$

Since $x - a > 0$, it follows that $f(x) - g(x) > 0$. Similarly, if $x - a < 0$, then $f(x) < g(x)$.

(b) An example is shown below.



18. (a) The position at time $t$ is

$$((v \cos \alpha)t, -16t^2 + (v \sin \alpha)t).$$

If $\cos \alpha = 0$, so that the cannon ball is shot straight up, then these points all lie on a straight line. If $\cos \alpha \neq 0$, then the set of all such points is equal to the set of all points

$$\left(t, -\frac{16t^2}{v \cos \alpha} + (\tan \alpha)t\right),$$

so the path of the cannon ball lies on the graph of

$$f(x) = \frac{-16x^2}{v \cos \alpha} + (\tan \alpha)x,$$

which is the graph of a parabola.

(b) The cannon ball hits the ground at time $t > 0$ when

$$0 = -16t^2 + (v \sin \alpha)t,$$

or $t = (v \sin \alpha)/16$ (of course we consider only $\alpha > 0$). It has then traveled a horizontal distance of

$$\begin{aligned} d(\alpha) &= (v \cos \alpha) \cdot \frac{v \sin \alpha}{16} \\ &= \frac{v^2 \sin \alpha \cos \alpha}{16}. \end{aligned}$$

Now $d(\alpha)$ is a maximum at that $\alpha$ for which

$$0 = d'(\alpha) = \frac{v^2}{16} [\cos^2 \alpha - \sin^2 \alpha],$$

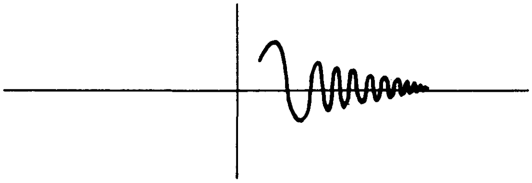
so $\tan \alpha = \pm 1$. Since only positive $\alpha$ are considered, $\alpha$ is a $45^\circ$ angle.

19. (a) Such a function f is pictured below. As an explicit example we can take $f(x) = (\sin x^2)/x$. Then $\lim_{x \to \infty} f(x) = 0$, but

$$f'(x) = \frac{2x^2 \sin x^2 - \sin x^2}{x^2}$$

$$= 2 \sin x^2 - \frac{\sin x^2}{x^2},$$

so $\lim_{x \to \infty} f'(x)$ does not exist.



(b) Let $\ell = \lim_{x \to \infty} f'(x)$. If $\ell > 0$, then there would be some N such that $|f'(x) - \ell| < |\ell|/2$ for $x > N$. This would imply that $f'(x) > |\ell|/2$. But that would imply, by the Mean Value Theorem, that

$$f(x) > f(N) + \frac{(x - N)|\ell|}{2} \qquad \text{for } x > N,$$

which would mean that $\lim_{x \to \infty} f(x)$ does not exist. Similarly, $\lim_{x \to \infty} f'(x)$ cannot be $< 0$.

(c) Let $\ell = \lim_{x \to \infty} f''(x)$. If $\ell > 0$, then, as in part (a), we have $\lim_{x \to \infty} f'(x) = \infty$. Another application of the Mean Value Theorem shows that $\lim_{x \to \infty} f(x) = \infty$, contradicting the hypothesis. Similarly, $\lim_{x \to \infty} f''(x)$ cannot be $< 0$.

(d) A similar proof shows that if $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} f^{(k)}(x)$ exist, then $\lim_{x\to\infty} f'(x) = \lim_{x\to\infty} f''(x) = \dots = \lim_{x\to\infty} f^{(k)}(x) = 0.$

20. If $g(x) \neq 0$ for all x in (a, b), then the function $h(x) = f(x)/g(x)$ is differentiable on (a, b), and by hypothesis

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} = 0.$$

This means that f/g is constant on (a, b), so $f = c\cdot g$ on (a, b) for some $c \neq 0$. Since f and g are continuous, it follows that $f(a) = c\cdot g(a)$, so $g(a) = 0$, contradicting the hypothesis.

21. We have

$$f'(x) = \lim_{y\to x} \frac{f(y) - f(x)}{y - x}.$$

Now

$$\left|\frac{f(y) - f(x)}{y - x}\right| \leq (x - y)^{n-1},$$

and $\lim_{y\to x} (x - y)^{n-1} = 0$, since $n - 1 > 0$. Consequently $f'(x) = 0$ for all x, so f is constant.

22. Let

$$f(x) = a_0x + \frac{a_1x^2}{2} + \dots + \frac{a_nx^{n+1}}{n + 1}.$$

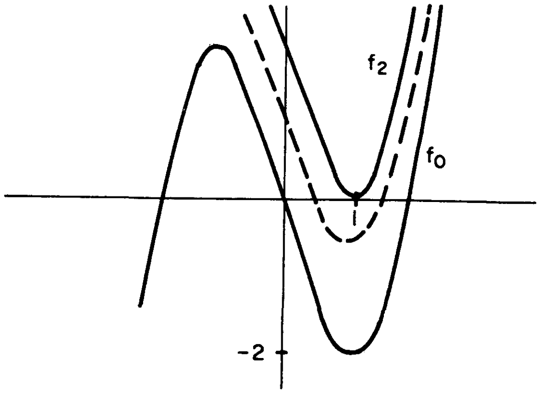
Then $f(0) = 0$, and $f(1) = 0$ by hypothesis. Rolle's Theorem implies that for some $x$ in $(0, 1)$ we have

$$0 = f'(x) = a_0 + a_1x + \dots + a_nx^n.$$

23. If $f_m(x_0) = f_m(x_1) = 0$ for $x_0 < x_1$ in $[0, 1]$, then $f_m'(x) = 0$ for some $x$ which is in $(x_0, x_1)$, and hence satisfies $0 < x < 1$. But

$$f_m'(x) = 3x^2 - 3 = 3(x^2 - 1),$$

so $f_m'(x) = 0$ only for $x = \pm 1$.



24. Problem 7-11 shows that there is at least one $x$. Suppose there were two, $x_0 < x_1$. The Mean Value Theorem, applied to $[x_0, x_1]$, would imply that

$$f'(x) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{x_1 - x_0}{x_1 - x_0} = 1$$

for some $x$ in $[x_0, x_1]$, contradicting the hypothesis.

25. Suppose that $f''(x) < 4$ for all $x$ in $[0, 1/2]$. Then, by the Mean Value Theorem, for all $x$ in $[0, 1/2]$ we have

$$\frac{f'(x) - f'(0)}{x - 0} = f''(x') \qquad \text{for some } x' \text{ in } [0, x]$$
$$< 4,$$

so $f'(x) < 4x$. Applying the Mean Value Theorem again, we have

$$\frac{f(x) - f(0)}{x - 0} = f'(x') \qquad \text{for some } x' \text{ in } (0, x)$$
$$< 4x' < 4x,$$

so $f(x) < 4x^2$. Consequently $f(1/2) < 1/2$.

The same sort of analysis can be applied to $f$ on $[1/2, 1]$ if $f''(x) < -4$ for all $x$ in $[0, 1/2]$. It is a little more convenient to introduce the function $g(x) = 1 - f(1 - x)$, which satisfies $g(0) = 0$ and $g''(x) = -f''(1 - x) < 4$ for $x$ in $[0, 1/2]$. As we have just shown

$$1/2 > g(1/2) = 1 - f(1/2),$$

so $f(1/2) > 1/2$, contradicting the result found before.

(The result of this problem can actually be strengthened: $|f''(x)| > 4$ for some $x$ in $[0, 1]$. To prove this we first note that we cannot have $f''(x) = 4$ for $0 < x < 1/2$ and also $f''(x) = -4$ for $1/2 < x \leq 1$, since this would imply that $f'(x) = 4x$ for $0 \leq x \leq 1/2$ and $f'(x) = -4x$ for $1/2 \leq x \leq 1$, in which case $f''(1/2)$ would not exist. On the other hand, if we have $f''(x) \leq 4$ for all $x$ in $(0, 1/2)$ but $f''(x) < 4$ for at least one $x$, then we have $f''(x) < 4x$ for at least one $x$, and consequently for all larger $x$ in $(0, 1/2)$, and therefore $f(x) < 4x^2$ for these $x$, so that $f(1/2) < 1/2$; if we also had $f''(x) \geq -4$ for all $x$ in $(1/2, 1)$, then $f(1/2) \geq 1/2$, a contradiction.)

26. If $g(x) = f(xy)$, then

$$\begin{aligned} g'(x) &= y \cdot f'(xy) \\ &= y \cdot \frac{1}{xy} = \frac{1}{x} = f'(x). \end{aligned}$$

So there is a number $c$ such that $g(x) = f(x) + c$ for all $x > 0$. Now

$$f(y) = g(1) = f(1) + c = c,$$

so $g(x) = f(x) + f(y)$.

27. Suppose $f(a) = f(b) = 0$. If $x$ is a local maximum point of $f$ on $[a, b]$, then $f'(x) = 0$ and $f''(x) \leq 0$; from the equation

$$f''(x) + f'(x)g(x) - f(x) = 0$$

we can conclude that $f(x) \leq 0$. Similarly, $f$ cannot have a negative local minimum on $(a, b)$.

28. If $f(x_i) = 0$ for $x_1 < x_2 < \dots < x_{n+1}$, then $f'(x) = 0$ for some $x$ in each of the $n$ intervals $(x_i, x_{i+1})$. Consequently $f''(x) = 0$ for $n - 1$ numbers $x$, etc. (In other words, we are all set up for a proof by induction.)

30. This is a trivial consequence of the Mean Value Theorem because if we define

$$g(x) = \begin{cases} \lim\limits_{y \to a^+} f(y), & x = a \\ f(x), & a < x < b \\ \lim\limits_{y \to b^-} f(y), & x = b, \end{cases}$$

then $g$ is continuous on $[a, b]$ and differentiable on $(a, b)$ and $g'(x) = f'(x)$ for $x$ in $(a, b)$, so there is some $x$ in $(a, b)$ with

$$f'(x) = g'(x) = \frac{g(b) - g(a)}{b - a}.$$

31. We have

$$[f(b) - f(a)]g'(x) = f'(x)[g(b) - g(a)].$$

If $g'(x) = 0$, then $f'(x)[g(b) - g(a)] = 0$. But this contradicts the assumption that $g(b) \neq g(a)$, and the fact that $f'(x) \neq 0$ (since $g'(x) = 0$).

32. Let

$$h(x) = f(x)g(b) + g(x)f(a) - f(x)g(x).$$

Then

$$h(a) = h(b) = f(a)g(b),$$

so by Rolle's Theorem there is some x in (a, b) with

$$0 = h'(x) = f'(x)g(b) + g'(x)f(a) - f'(x)g(x) - f(x)g'(x),$$

or

$$f'(x)[g(b) - g(x)] = g'(x)[f(x) - f(a)].$$

Since $g'(x) \neq 0$ for all x in (a, b), we also have $g(b) \neq g(x)$ for x in (a, b) (otherwise Rolle's Theorem, applied to the interval [x, b] would imply that $g'(x) = 0$ for some x' in (x, b).)

35. Since $g(0) = 0$, and g is continuous at 0, we have $\lim_{x \to 0} g(x) = 0$. Therefore, by l'Hopital's Rule

$$f'(0) = \lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} \frac{g(x)}{x^2}$$

$$= \lim_{x \to 0} \frac{g'(x)}{2x} = \lim_{x \to 0} \frac{g'(x) - g'(0)}{2x} = g''(0) = \frac{17}{2}.$$

(The limit $\lim_{x \to 0} g'(x)/2x$ could also be found by l'Hopital's Rule.)

36. (a) Use exactly the same proof as for l'Hopital's Rule, but consider only x in $(a, a + \delta)$ or in $(a - \delta, a)$, respectively.

(b) Again the proof of l'Hopital's Rule will work, almost verbatim. (It is tempting to apply l'Hopital's rule to g/f: Since $\lim_{x \to a} g'(x)/f'(x) = 0$, it follows that $\lim_{x \to a} g(x)/f(x) = 0$. Unfortunately, this implies only that $\lim_{x \to a} |f(x)/g(x)| = \infty$.)

(c) Since $\lim_{x\to 0^+} f(1/x) = \lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to 0^+} g(1/x) = \lim_{x\to\infty} g(x) = 0$, part (a) implies that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{f(1/x)}{g(1/x)} = \lim_{x\to 0^+} \frac{-(1/x^2)f'(1/x)}{-(1/x^2)g'(1/x)}$$

$$= \lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \ell.$$

(d) Similar to part (c), using the case $x\to a^+$ of part (b) instead of part (a).

37. (a) For any $\epsilon > 0$ there is some a such that

$$\left|\frac{f'(x)}{g'(x)} - \ell\right| < \epsilon \qquad \text{for} \quad x \geq a.$$

This means, in particular, that $g'(x) \neq 0$ for $x > a$; it follows that $g(x) - g(a) \neq 0$ for $x > a$ (by Rolle's Theorem). Therefore the Cauchy Mean Value Theorem can be written in the form

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(x')}{g'(x')} \qquad \text{for some } x' \text{ in } (a, x).$$

Since $x' > a$, the desired inequality follows.

(b) We have

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} \cdot \frac{f(x)}{f(x) - f(a)} \cdot \frac{g(x) - g(a)}{g(x)},$$

where $f(x) - f(a) \neq 0$, $g(x) \neq 0$ for large enough x, since $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$. These limits also imply that

$$\lim_{x\to\infty} \frac{f(x)}{f(x) - f(a)} = \lim_{x\to\infty} \frac{g(x) - g(a)}{g(x)} = 1.$$

It follows that $f(x)/g(x)$ can be made as close as desired to $[f(x) - f(a)]/[g(x) - g(a)]$ by choosing x large enough. Together with part (a), this shows that

$$\left|\frac{f(x)}{g(x)} - \ell\right| < 2\epsilon \qquad \text{for sufficiently large } x.$$

38. One other form of l'Hopital's Rule will be used in later problems: If $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$ and $\lim_{x\to\infty} f'(x)/g'(x) = \infty$, then $\lim_{x\to\infty} f(x)/g(x) = \infty$. To prove this, apply Problem 37 to g/f: Since $\lim_{x\to\infty} g'(x)/f'(x) = 0$, we have $\lim_{x\to\infty} g(x)/f(x) = 0$. This implies (as we remarked in the solution to Problem 36) that $\lim_{x\to\infty} |f(x)/g(x)| = \infty$. Since $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$, we can conclude that $\lim_{x\to\infty} f(x)/g(x) = \infty$.

39. (a) Since a is a minimum point for f on [a, b], for all sufficiently small $h > 0$ we have

$$\frac{f(a + h) - f(a)}{h} \geq 0;$$

this implies that $f'(a) \geq 0$. The proof that $f'(b) \leq 0$ is similar.

(b) Part (a) shows that we cannot have the minimum of $f$ at $a$ or at $b$, since we are assuming that $f'(a) < 0$ and $f'(b) > 0$. So the minimum occurs at some point $x$ in $(a, b)$. Then $f'(x) = 0$.

(c) Let $g(x) = f(x) - cx$. Then $g'(a) = f'(a) - c < 0$ and $g'(b) = f'(b) - c > 0$. So by part (b), $0 = g'(x) = f'(x) - c$ for some $x$ in $(a, b)$.

40. If $f(a) \neq 0$, then continuity of $f$ implies that $f = |f|$ or $f = -|f|$ in some interval around $a$, so $f$ is differentiable at $a$. If $f(a) = 0$, then $a$ is a minimum point for $|f|$, so $|f|'(a) = 0$. This means that

$$\begin{aligned} 0 &= \lim_{h \to 0} \frac{|f(a+h)| - |f(a)|}{h} \\ &= \lim_{h \to 0} \frac{|f(a+h)|}{h}. \end{aligned}$$

This equation also says that $f'(a) = 0$.

41. (a) If $f(x_0) = 0$ for some $x_0 \neq 0$, then Rolle's Theorem would imply that

$$0 = f'(x) = nx^{n-1} - n(x+y)^{n-1} \qquad \text{for some } x \text{ in } (0, x_0) \text{ or } (x_0, 0).$$

But this means that $x^{n-1} = (x+y)^{n-1}$ for $y \neq 0$, which is impossible, since $g(x) = x^{n-1}$ is increasing $(n - 1$ is odd$)$.

(b) Let $f(x) = x^n + y^n - (x+y)^n$. Then $f(0) = f(-y) = 0$. If $f$ were zero at three points $a < b < c$, then Rolle's Theorem could be applied to [a, b] and [b, c] to prove that there are two numbers x with

$$0 = f'(x) = nx^{n-1} - n(x + y)^{n-1};$$

but this equation holds only for $x = -(x + y)$, or $x = y/2$ (Problem 1-7).

42. The tangent line through $(a, a^n)$ is the graph of

$$\begin{aligned} g(x) &= na^{n-1}(x - a) + a^n \\ &= na^{n-1}x + (1 - n)a^n. \end{aligned}$$

If $g(x_0) = f(x_0)$ for some $x_0 \neq a$, then Rolle's Theorem may be applied to $g - f$ on the interval $[a, x_0]$, or $[x_0, a]$:

$$0 = g'(x) - f'(x) = na^{n-1} - nx^{n-1} \qquad \text{for some } x \text{ in } (a, x_0) \text{ or } (x_0, a).$$

This is impossible, since $x \neq a$ and $n - 1$ is odd, so $a^{n-1} \neq x^{n-1}$.

43. The tangent line through $(a, f(a))$ is the graph of

$$\begin{aligned} g(x) &= f'(a)(x - a) + f(a) \\ &= f'(x)x + f(a) - af'(a). \end{aligned}$$

If $g(x_0) = f(x_0)$ for some $x_0 \neq a$, then

$$0 = g'(x) - f'(x) = f'(a) - f'(x) \qquad \text{for some } x \text{ in } (a, x_0) \text{ or } (x_0, a).$$

This is impossible, since $f'$ is increasing.

44. Since

$$g'(x) = \frac{xf'(x) - f(x)}{x^2},$$

it suffices to show that

$$xf'(x) - f(x) > 0,$$

or

$$f'(x) > \frac{f(x)}{x} \qquad \text{for } x > 0.$$

Now the Mean Value Theorem, applied to f on [0, x], shows that

$$\begin{aligned} \frac{f(x)}{x} = \frac{f(x) - f(0)}{x - 0} &= f'(x') &&\text{for some } x' \text{ in } [0, x] \\ &< f'(x), &&\text{since } f' \text{ is increasing.} \end{aligned}$$

45. Let $g(x) = (1 + x)^n - (1 + nx)$. Then $g(0) = 0$, but

$$g'(x) = n(1 + x)^{n-1} - n.$$

Since $n - 1 \geq 0$ this means that

$$\begin{aligned} g'(x) &< 0 &&\text{for } -1 < x < 0, \\ &> 0 &&\text{for } x > 0. \end{aligned}$$

Thus $g(x) > 0$ for $-1 < x < 0$ and $0 < x$.

46. (a) 0 is actually a minimum on all of **R**, since $f(0) = 0$ and $f(x) \geq 0$ for all x.

(b)

$$f'(0) = \lim_{h \to 0} \frac{h^4 \sin^2(1/h)}{h} = 0,$$

and

$$f'(h) = 4h^3 \sin^2(1/h) - 2h^2 \sin(1/h) \cos(1/h) \qquad \text{for } h \neq 0.$$

So

$$\begin{aligned} f''(0) &= \lim_{h \to 0} \frac{4h^3 \sin^2(1/h) - 2h^2 \sin(1/h) \cos(1/h)}{h} \\ &= 0. \end{aligned}$$

47. (a) Since $f'$ is continuous, $f'(x) > 0$ for all $x$ in some interval around $a$, so $f$ is increasing in this interval.

(b) We have

$$g'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}.$$

So $g'(x) = 1$ when $\cos 1/x = 1$ (and consequently $\sin 1/x = 0$), and $g'(x) = -1$ when $\cos 1/x = -1$.

(c) We have $f'(x) = \alpha + g'(x)$, so $f'(x) > 0$ when $g'(x) = 1$, and $f'(x) < 0$ when $g'(x) = -1$.

48. (a) We have

$$g(y) = \frac{2 \sin y}{y} - \cos y,$$

so

$$g'(y) = \frac{2y \cos y - 2 \sin y}{y^2} + \sin y.$$

So if $g'(y) = 0$, then

$$0 = 2y \cos y - 2 \sin y + y^2 \sin y,$$

or

$$(1) \qquad \cos y = \frac{2 \sin y - y^2 \sin y}{2y} = \sin y \left(\frac{2 - y^2}{2y}\right).$$

Hence

$$(2) \qquad g(y) = \frac{2 \sin y}{y} - \sin y \left(\frac{2 - y^2}{2y}\right)$$

$$= \sin y\left(\frac{2}{y} - \frac{2 - y^2}{2y}\right)$$

$$= \sin y \left(\frac{2 + y^2}{2y}\right).$$

(b) Moreover, from (1) we have

$$1 - \sin^2 y = \cos^2 y = \sin^2 y \left(\frac{2 - y^2}{2y}\right)^2,$$

so

$$\sin^2 y = \frac{1}{1 + \left(\frac{2 - y^2}{2y}\right)^2} = \frac{4y^2}{4 + y^4},$$

so, by (2),

$$|g(y)| = |\sin y| \cdot \left|\frac{2+y^2}{2y}\right|$$

$$= \frac{|2y|}{\sqrt{4+y^4}} \cdot \frac{2+y^2}{|2y|} = \frac{2+y^2}{\sqrt{4+y^4}}.$$

(c) We have

$$f'(x) = 1 + g(1/x).$$

Now we clearly have $g(y) < 0$ for arbitrarily large $y$ (since $g(y)$ is practically $-\cos y$ for large $y$), so for arbitrarily large $y$ we have

$$g(y) < -\frac{2+y^2}{\sqrt{4+y^4}} < -1$$

by part (b). Thus $f'(x) < 0$ for arbitrarily small $x$, while we also have $f'(x) > 0$ for arbitrarily small $x$.

(d) We have

$$f'(x) = \alpha + g(1/x).$$

For sufficiently large $y$ we have $g(y) > -\alpha$. So for sufficiently small $x$ we have $f'(x) > 0$.

49. (a) If the minimum of $f$ on $[b, 1]$ occurred at some $c$ with $b < c \le 1$, then $f$ would clearly not be increasing at $c$, since we would have $f(x) \ge f(c)$ for all $x < c$ sufficiently close to $c$. Now if $0 \le a < b \le 1$, then the

minimum of f on [a, 1] is a, so $f(a) \leq f(b)$. To obtain the strict inequality $f(a) < f(b)$, pick some $a'$ with $a < a' < b$ such that $f(a') > f(a)$ (this is possible since f is increasing at a); then $f(a) < f(a') \leq f(b)$.

(b) Let $\alpha = \sup S_b$. If $b \leq y < \alpha$, then there is some x in $S_b$ with $y < x$. Therefore $f(y) \geq f(b)$. Moreover, since f is increasing at $\alpha$, we have $f(\alpha) > f(x)$ for $x < \alpha$ sufficiently close to $\alpha$, so $f(\alpha) > f(b)$. This shows that $\alpha$ is actually in $\sup S_b$. Now if $\alpha < 1$ there would be a $\delta > 0$ such that $f(x) > f(\alpha)$ for $\alpha < x < \alpha + \delta$. This shows that all such x are in $S_b$, contradicting the fact that $\alpha = \sup S_b$. So $\alpha = \sup S_b = 1$. So $f(y) \geq f(b)$ for all $y \geq b$.

(c) For sufficiently small h we have

$$\begin{aligned} f(a+h) &> f(a) \qquad \text{if } h > 0, \\ f(a+h) &< f(a) \qquad \text{if } h < 0. \end{aligned}$$

This implies that

$$\frac{f(a+h)-f(a)}{h} > 0,$$

which implies that

$$f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h} \geq 0.$$

(d) Since $f'(a) > 0$, for sufficiently small $h$ we have

$$\frac{f(a + h) - f(a)}{h} > 0.$$

This implies that $f(a + h) > f(a)$ for $h > 0$ and $f(a + h) < f(a)$ for $h < 0$.

(e) Part (d) implies that $f$ is increasing at $a$ for all $a$ in $[0, 1]$, so part (a) implies that $f$ is increasing on $[0, 1]$.

(f) If $\epsilon > 0$, then $g'(a) = f'(a) + \epsilon = \epsilon > 0$ for all $a$ in $[0, 1]$, so $g$ is increasing on $[0, 1]$ by part (e), so $f(1) + \epsilon > f(0)$, or $f(1) - f(0) > -\epsilon$. Similarly, $h$ is increasing on $[0, 1]$, so $\epsilon - f(1) > - f(0)$, or $f(1) - f(0) < \epsilon$. Thus $|f(1) - f(0)| < \epsilon$. Since this is true for all $\epsilon > 0$, it follows that $f(1) = f(0)$. (Of course, the same argument, applied to $[a, b]$, for $0 \leq a < b \leq 1$, shows that $f(a) = f(b)$.)

50. Let $\alpha = \sup A$, where $A = \{x \text{ in } [x_0, b]: f(y) = f(x_0) \text{ for all } y \text{ in } [x_0, x]\}$. It is clear that $f(\alpha) = f(x_0)$, since $f$ is continuous and there are $x$ arbitrarily close to $\alpha$ with $f(x) = f(x_0)$. It is also easy to see that $f(y) = f(x_0)$ for all $y < \alpha$. So $\alpha$ is in $A$. If $\alpha < 1$, then there is a $\delta > 0$ such that $f(x) \leq f(\alpha)$ for $\alpha < x < \alpha + \delta$, since $\alpha$ is a local maximum point. But

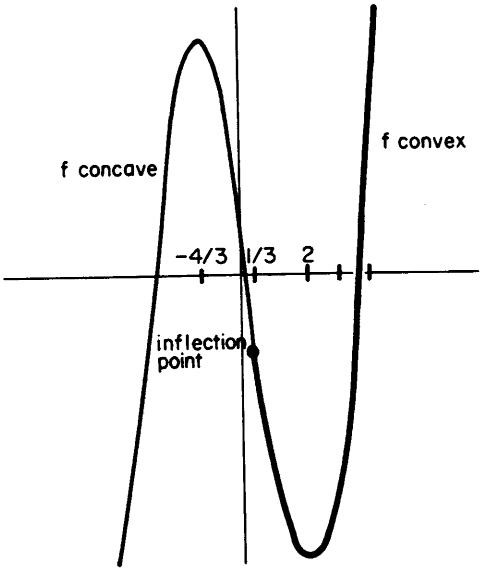
$f(\alpha) = f(x_0)$, the minimum value of $f$ on $[a, b]$. So $f(x) = f(\alpha)$ for $\alpha < x < \alpha + \delta$. This implies that all such $x$ are in A, a contradiction. So $f(x) = f(x_0)$ for all $x$ in $[x_0, b]$. A similar proof shows that $f(x) = f(x_0)$ for all $x$ in $[a, x_0]$.

51. (a) The local strict maximum points are the rational numbers.

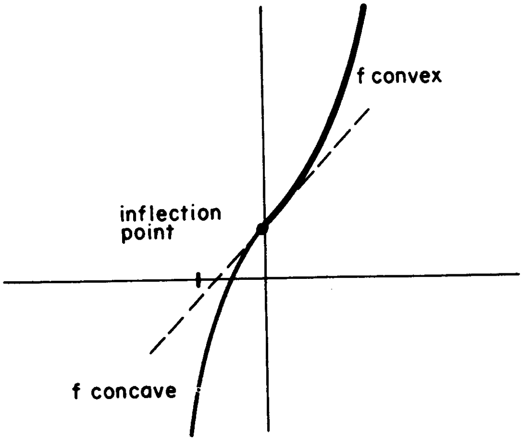
(b) Let $x$ be a point in all intervals $I_n = [a_n, b_n]$. Since the points $x_n$ are chosen to be distinct, $x = x_n$ for at most one $n$. Since $x$ is a local strict maximum point, there is a $\delta > 0$ such that $x$ is a strict maximum point for $f$ on $(x - \delta, x + \delta)$. But $I_n$ is contained in $(-x - \delta, x + \delta)$ for all sufficiently large $n$; choose such an $n$ for which $x \neq x_n$. Then $f(x) > f(x_n)$, since $I_n$ is contained in $(x - \delta, x + \delta)$, while $f(x_n) > f(x)$, since $x$ is in $I_n$.

# APPENDIX

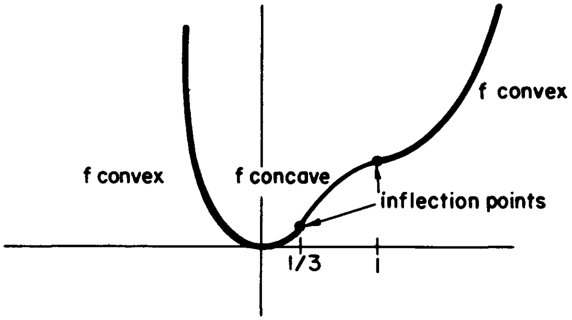
1. (i) $f''(x) = 6x - 2 > 0$ for $x > 1/3$.



(ii) $f''(x) = 20x^3 + 1 > 0$ for $x > -1/\sqrt[3]{20}$.



(iii) $f''(x) = 36x^2 - 48x + 12 = 12(3x^2 - 4x + 1) = 12(3x - 1)(x - 1) > 0$ for $x < 1/3$ or $x > 1$.



(iv) We have

$$f''(x) = \frac{-(x^5 + x + 1)^2\ 20x^3 + (5x^4 + 1)\ 2(x^5 + x + 1)(5x^4 + 1)}{(x^5 + x + 1)^4}$$

$$= \frac{2(x^5 + x + 1)[(5x^4 + 1)^2 - 10x^3(x^5 + x + 1)]}{(x^5 + x + 1)^4}$$

$$= \frac{2}{(x^5 + x + 1)^3}[15x^8 - 10x^3 + 1].$$

To determine the sign of $f''(x)$ it suffices to determine the sign of

$$g(x) = 15x^8 - 10x^3 + 1.$$

Now

$$g'(x) = 120x^7 - 30x^2 = 30x^2(4x^5 - 1).$$

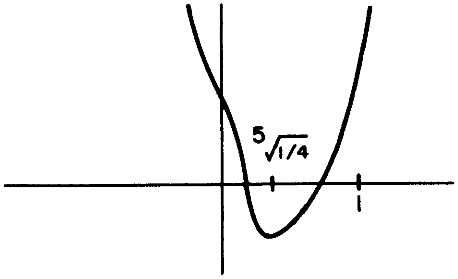
So $g'(x) = 0$ for $x = 0$ or $x = \sqrt[5]{1/4}$. We have $g(0) = 1$ and

$$g(\sqrt[5]{1/4}) = (\sqrt[5]{1/4})^3[15\cdot\tfrac{1}{4} - 10] + 1$$

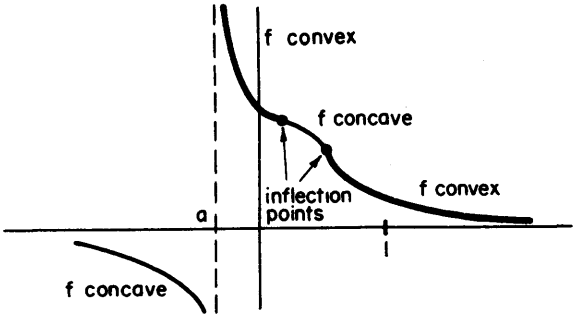
$$= (\sqrt[5]{1/4})^3\left(\frac{-25}{4}\right) + 1$$

$$< 0,$$

since $\sqrt[5]{1/64} > 4/25$. So g attains its (negative) minimum value at $\sqrt[5]{1/4}$. Moreover, since $h(x) = 4x^5 - 1$ is increasing, $g'(x) > 0$ for $x > \sqrt[5]{1/4}$ and $g'(x) < 0$ for $x < \sqrt[5]{1/4}$. So g is decreasing on $(-\infty, \sqrt[5]{1/4}]$ and increasing on $[\sqrt[5]{1/4}, \infty)$.



Consequently, g has two zeros, both in [0, 1], since $g(1) > 0$. It follows that if a is the unique root of $x^5 + x + 1 = 0$, then $f''(x) < 0$ for $x < a$, but $f''(x) > 0$ for all $x > a$ except those x in a certain interval contained in (0, 1). Thus the graph of f is convex on $(a, \infty)$, except for a bump lying over some interval contained in (0, 1).



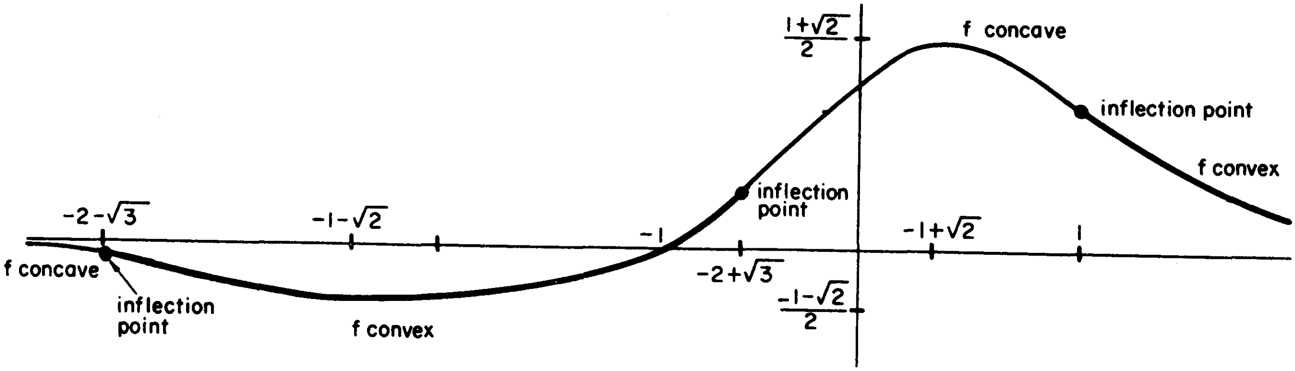
(v) We have

$$f''(x) = \frac{(x^2 + 1)^2(-2 - 2x) - (1 - 2x - x^2)\ 2(x^2 + 1)\ 2x}{(x^2 + 1)^4}$$

$$= \frac{2}{(x^2 + 1)^3}\,[x^3 + 3x^2 - 3x - 1] = \frac{2}{(x^2 + 1)^3}\,(x - 1)(x^2 + 4x + 1)$$

$$= \frac{2}{(x^2 + 1)^3}\,(x - 1)(x - [-2 + \sqrt{3}])(x - [-2 - \sqrt{3}]),$$

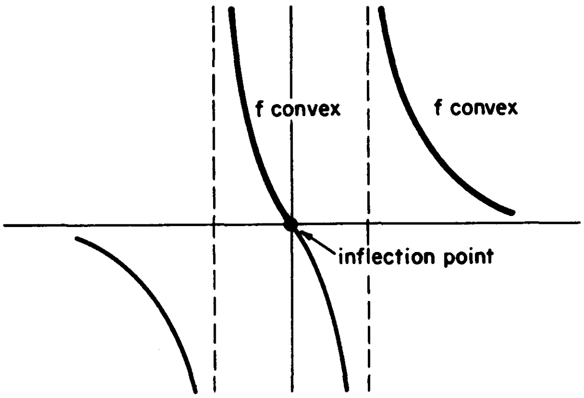
so $f''(x) > 0$ for $-2 - \sqrt{3} < x < -2 + \sqrt{3}$ and $x > 1$.



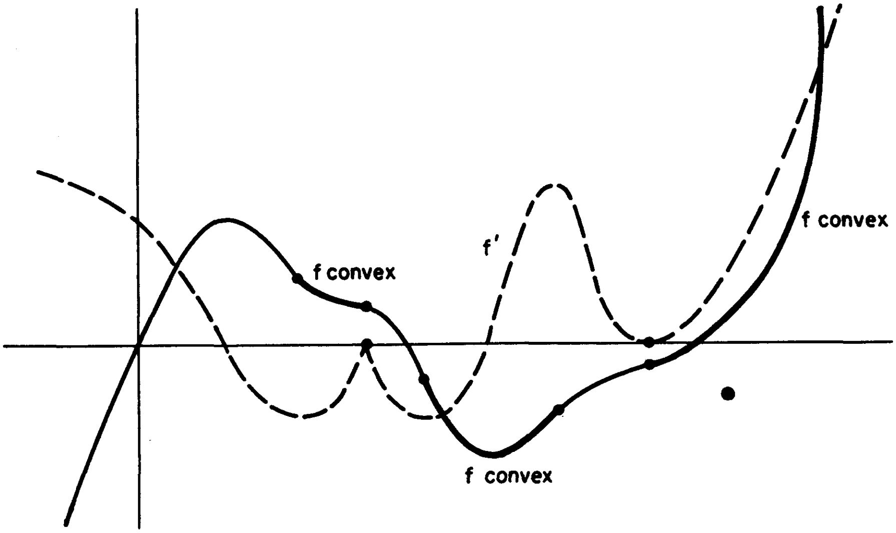
(vi) $$f''(x) = \frac{(x^2 - 1)^2\ (-2x) + (1 + x^2)\ 2(x^2 - 1)\ 2x}{(x^2 - 1)^4}$$

$$= \frac{2x}{(x^2 - 1)^3}\,[x^2 + 3],$$

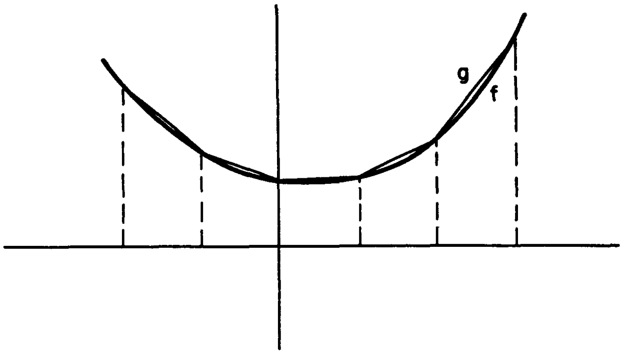
so $f''(x) > 0$ for $x > 1$ and $-1 < x < 0$.



2. If f(0) = 0, the graph looks like the following.



3. Two such functions are shown below.



4. The points in $(x, y)$ are precisely those of the form $tx + (1 - t)y$ for $0 < t < 1$. In fact, if $0 < t < 1$, then

$$x = tx + (1 - t)x < tx + (1 - t)y < ty + (1 - t)y = y,$$

and conversely, if $x < z < y$, then

$$z = tx + (1 - t)y \qquad \text{for } 0 < t = \frac{y - z}{y - x} < 1.$$

Definition 2 thus shows that $f$ is convex if and only if

$$\frac{f(tx + (1 - t)y) - f(x)}{tx + (1 - t)y - x} < \frac{f(y) - f(x)}{y - x},$$

which is equivalent to

$$f(tx + (1 - t)y) < tf(x) + (1 - t)f(y).$$

5. If $a < x < b$, then $f(a) < f(x) < f(b)$, so

$$\frac{g(f(x)) - g(f(a))}{f(x) - f(a)} < \frac{g(f(b)) - g(f(a))}{f(b) - f(a)},$$

and consequently,

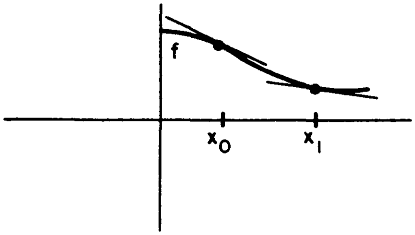
$$\begin{aligned}\frac{g(f(x)) - g(f(a))}{x - a} &= \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} \cdot \frac{f(x) - f(a)}{x - a}\\ &< \frac{g(f(b)) - g(f(a))}{f(b) - f(a)} \cdot \frac{f(b) - f(a)}{b - a}\\ &= \frac{g(f(b)) - g(f(a))}{b - a}.\end{aligned}$$

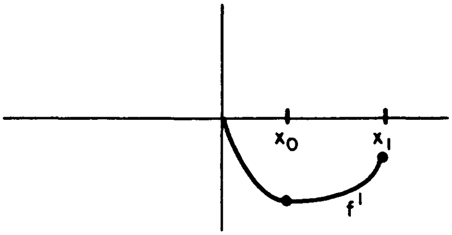
(If $f$ and $g'$ are differentiable, a simpler proof can be given: $(g\circ f)' = (g'\circ f)\cdot f'$ is increasing because $g'\circ f$ is increasing (since $g'$ and $f$ are), and $f'$ is increasing and positive.)

6. Choose $x > 0$ so that $f(x) < f(0)$. The Mean Value Theorem implies that there is some $x_0$ in $(0, x)$ with $f'(x_0) < 0$. If we had $f'(y) \leq f'(x_0)$ for all $y \geq x_0$, then for all $x > x_0$ we would have

$$f(x) - f(x_0) \leq f'(x_0)(x - x_0),$$

which would imply that $f(x)$ is eventually negative (since $f'(x_0) < 0$). Therefore $f'(x_1) > f'(x_0)$ for some $x_1 > x_0$.





This implies that the minimum of $f'$ on $[0, x_1]$ occurs at some $x$ in $(0, x_1)$. Then $f''(x) = 0$.

7. (a) This follows from Problem 4 with $t = 1/2$.

(b) The assertion is true for $n = 1$, i.e., $k = 1/2$. Suppose that for some $n$ it is true for all $x$ and $y$. If $k = m/2^{n+1}$ is in lowest terms, then $k$ is odd. Consequently $k_1 = (m - 1)/2^{n+1}$ and $k_2 = (m + 1)/2^{n+1}$ can be expressed in the form $a/2^n$, so the assertion is true for $k_1$ and $k_2$. Notice also that $k = (k_1 + k_2)/2$. From the result for $k_1$ and $k_2$, and the assertion for $n = 1$ applied to $x' = k_1x + (1 - k_1)y$ and $y' = k_2x + (1 - k_2)y$, we obtain

$$f(kx + (1 - k)y) = f\left(\frac{x' + y'}{2}\right) < \frac{f(x')}{2} + \frac{f(y')}{2}$$
$$< \frac{k_1f(x) + (1 - k_1)f(y)}{2} + \frac{k_2f(y) + (1 - k_2)f(y)}{2}$$
$$= kf(x) + (1 - k)f(y).$$

(c) Let $0 < t < 1$. For any $\epsilon > 0$ there is a number $k$ of the form $m/2^n$ which is so close to $t$ that

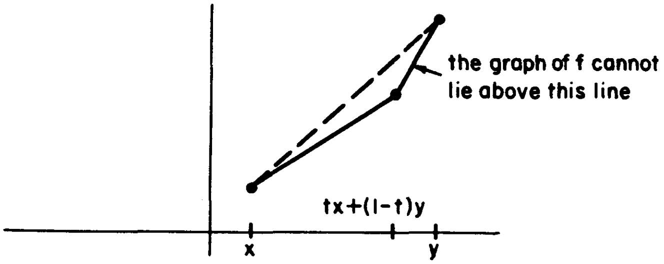
$$|f(kx + (1 - k)y) - f(tx + (1 - t)y)| < \epsilon,$$

$$|[kf(x) + (1 - k)f(y)] - [tf(x) + (1 - t)f(y)]| < \epsilon.$$

Then

$$\begin{aligned} f(tx + (1 - t)y) &< f(kx + (1 - k)y) + \epsilon \\ &< kf(x) + (1 - k)f(y) + \epsilon \\ &< tf(x) + (1 - t)f(y) + 2\epsilon. \end{aligned}$$

Thus $f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$. The following diagram shows that if strict inequality holds for even one $t$, then it holds for all $t$ (by applying the weak inequality to $x$ and $tx + (1 - t)y$ or to $tx + (1 - t)y$ and $y$).



But we have strict equality for $t$ of the form $m/2^n$, so we must have strict inequality for all $t$.

8. (a) Let $x_\alpha$ and $x_\beta$ be the smallest and largest $x_i$. Then

$$x_\alpha = \sum_{i=1}^{n} p_i x_\alpha \leq \sum_{i=1}^{n} p_i x_i \leq \sum_{i=1}^{n} p_i x_\beta = x_\beta.$$

(b) Part (a), applied to $p_1/t,\dots,p_{n-1}/t$, shows that $(1/t)\sum_{i=1}^{n-1} p_ix_i$ lies between the smallest and largest of $x_1,\dots,x_{n-1}$, so it certainly lies between the smallest and largest of $x_1,\dots,x_n$.

(c) Jensen's Theorem is true for $n = 1$. Suppose it is true for $n - 1$. Then by Problem 4 we have, since $p_n = 1 - t$,

$$\begin{aligned} f\left(\sum_{i=1}^{n} p_ix_i\right) &= f\left(t\cdot(1/t)\sum_{i=1}^{n-1} p_ix_i + (1 - t)x_n\right) \\ &\le t\, f\left(\sum_{i=1}^{n-1} (p_i/t)x_i\right) + (1 - t)f(x_n) \\ &\le t \sum_{i=1}^{n-1} \frac{p_i}{t} f(x_i) + p_nf(x_n) \\ &= \sum_{i=1}^{n} p_if(x_i). \end{aligned}$$

(The same sort of proof shows that strict inequality holds for $n > 1$ (begin by checking that strict inequality holds for $n = 2$).)

9. (a) As the proof of Theorem 1 shows, $[f(a + h) - f(a)]/h$ is decreasing as $h \to 0^+$, so

$$f_+'(a) = \lim_{h\to 0^+} \frac{f(a + h) - f(a)}{h} = \inf\left\{\frac{f(a + h) - f(a)}{h} : h > 0\right\}.$$

This inf exists because each quotient $[f(a + h) - f(a)]/h$ for $h > 0$ is greater than any one such quotient for $h < 0$. Similarly,

$$f_-'(a) = \sup\left\{\frac{f(a+h) - f(a)}{h}: h < 0\right\}.$$

The relation $f_-'(a) \le f_+'(a)$ is obvious from the previous considerations. The functions $f_+'$ and $f_-'$ are increasing, because if $a < b$, then (as in the proof of Theorem 1; see Figure 6) we have

$$f_-'(a) \le f_+'(a) < \frac{f(a + (b - a)) - f(a)}{b - a} = \frac{f(b + (a - b)) - f(b)}{a - b} < f_-'(b) \le f_+'(b).$$

(b) If $b < a$, then as in part (a) (with a and b interchanged) we have

$$f_+'(b) < f_-'(a) \le f_+'(a).$$

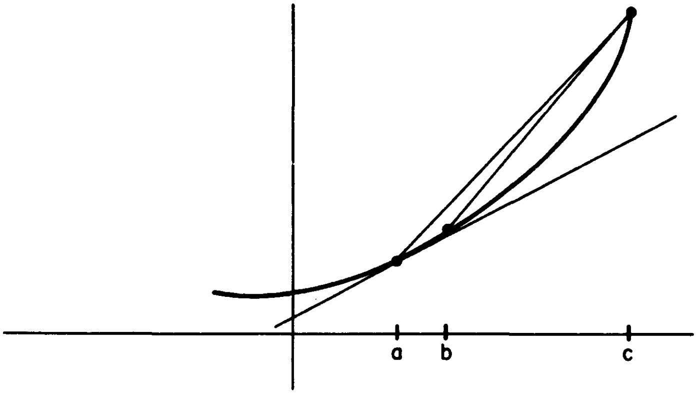
If $f_+'$ is continuous at a, then $\lim_{b \to a} f_+'(b) = f_+'(a)$, so we must have $f_-'(a) = f_+'(a)$.

To prove the converse, we first show that $f_+'$ will always be continuous on the right, i.e.,

$$\lim_{b \to a^+} f_+'(b) = f_+'(a).$$

In fact, for any $\epsilon > 0$ we can choose $c > a$ so that

$$\frac{f(c) - f(a)}{c - a} < f_+'(a) + \epsilon.$$



Since $f_+'(a)$ exists, f satisfies $\lim_{b \to a^+} f(b) = f(a)$ (as a matter of fact, f is continuous at a even if $f_+'(a)$ does not exist; see Problem 11). So we can choose $b > a$ close to a with f(b) as close to f(a) as desired. Thus we can choose $b > a$ so that

$$\frac{f(c) - f(b)}{c - b} < \frac{f(c) - f(a)}{c - a} + \epsilon.$$

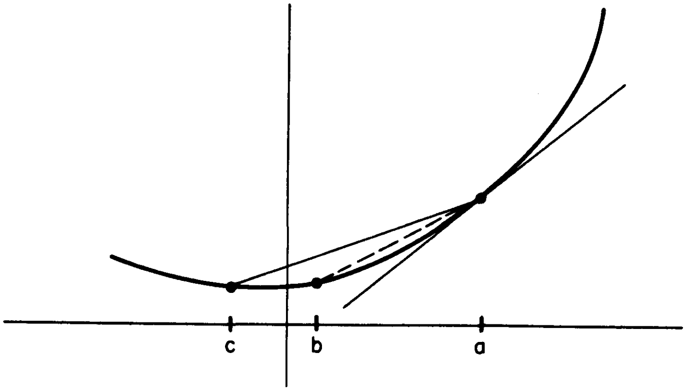
Therefore

$$\begin{aligned} f_+'(a) < f_+'(b) &< \frac{f(c) - f(b)}{c - b} \\ &< \frac{f(c) - f(a)}{c - a} + \epsilon \\ &< f_+'(a) + 2\epsilon. \end{aligned}$$

This shows that $f_+'$ is continuous on the right.

It remains to show that if $f_+'(a) = f_-'(a)$, then $f_+'$ is continuous on the left at a. Given $\epsilon > 0$, choose $c < a$ so that

$$f_+'(a) - \epsilon = f_-'(a) - \epsilon < \frac{f(a) - f(c)}{c - a}.$$

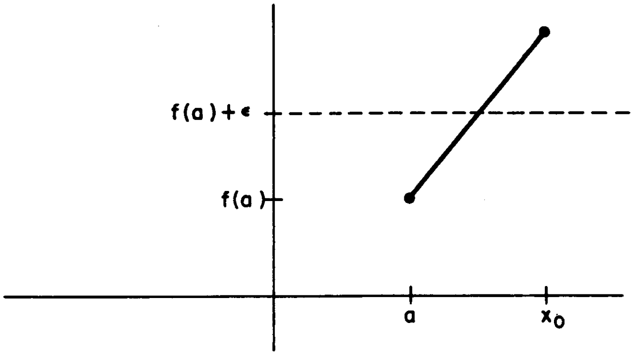


Then if $c < b < a$, the secant line through $(b, f(b))$ and $(a, f(a))$ lies between the tangent line at a and the secant line through $(c, f(c))$ and $(a, f(a))$, i.e.,

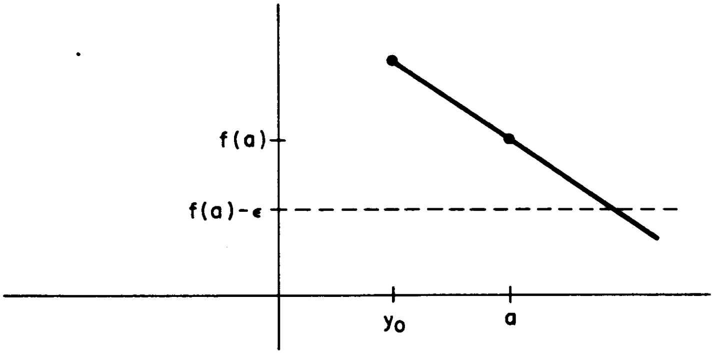
$$f_+'(a) - \epsilon < \frac{f(a) - f(c)}{c - a} < \frac{f(a) - f(b)}{a - b} < f_+'(b) < f_+'(a).$$

This shows that $\lim_{b \to a^-} f_+'(b) = f_+'(a)$.

10. Let $\epsilon > 0$. Pick some $x_0 > a$. Notice that no matter what value $f(x_0)$ may have, the line segment between $(a, f(a))$ and $(x_0, f(x_0))$ eventually lies below the horizontal line at height $f(a) + \epsilon$. Since the graph of f must lie below this line on $(a, x_0)$, this shows that $f(x) < f(a) + \epsilon$ for all $x > a$ sufficiently close to a. A similar argument works for all $x < a$ sufficiently close to a.



It remains to show that $f(x) > f(a) - \epsilon$ for all x sufficiently close to a. If $f(x) \geq f(a)$ for all x there is nothing to prove, so suppose that $f(x_0) < f(a)$ for some $x_0$ with $x_0 > a$, say. Then we must have $f(x) > f(a)$ for all $y < a$, because of convexity, so all $y < a$ certainly satisfy $f(y) > f(a) - \epsilon$. Moreover, if we pick some $y_0 < a$, then the line segment between $(y_0, f(y_0))$ and $(a, f(a))$ lies above the horizontal line at height $f(a) - \epsilon$ in some interval to the right of a. Since the graph of f must lie above this line to the right of a, it follows that $f(x) > f(a) - \epsilon$ for all $x > a$ sufficiently close to a.

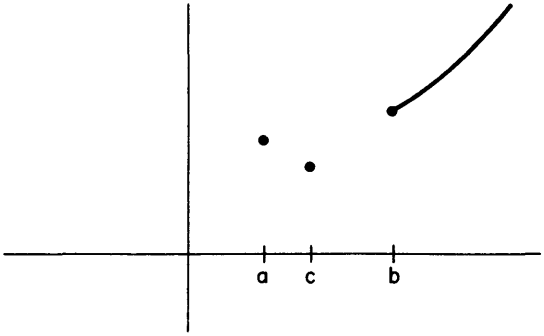


11. Lemma. Suppose $f$ is convex on $\mathbf{R}$ and $a < b$. If $f(a) < f(b)$, then $f$ is increasing on $[b, \infty)$, and if $f(a) > f(b)$, then $f$ is decreasing on $(-\infty, a]$.

Proof. Consider the case $f(a) < f(b)$ (the proof in the other case is similar, or one can apply this first case to $g(x) = f(-x)$.)

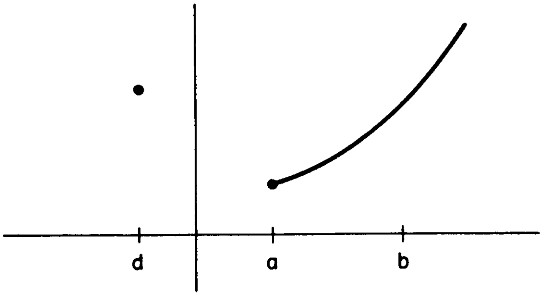
If $b < d$, then the definition of convexity shows immediately that we cannot have $f(d) \leq f(b)$. Moreover, if $b < d_1 < d_2$, then the same argument shows (since we now know that $f(b) < f(d_1)$) that $f(d_1) < f(d_2)$. Thus $f$ is increasing on $[b, \infty)$.

With the aid of this lemma we can now prove the theorem. Since $f$ is not constant, there is some $a < b$ with $f(a) \neq f(b)$. We consider only the case $f(a) < f(b)$. We already know from the lemma that $f$ is increasing on $[b, \infty)$. Suppose now that the minimum of $f$ on $[a, b]$ occurs at some $c$ in $(a, b)$.



Then f is decreasing on $(-\infty, a]$ by the lemma. Moreover, if $a'$ is any number with $a < a' < c$, then we must have $f(a') > f(c)$ (if we had $f(a') = f(c)$, then $f(x) < f(c)$ for x in $(a', c)$, contradicting the fact that c is the minimum point). So the lemma also implies that f is decreasing on $(-\infty, a']$ for all such $a'$. This shows that f is decreasing on $(-\infty, c]$. Similarly, f is increasing on $[c, \infty)$.

On the other hand, suppose that the minimum of f on $[a, b]$ occurs at a. The same sort of reasoning as before shows that f is increasing on $[a, \infty)$. There are then two possibilities. It may happen that $f(d) > f(a)$ for some $d < a$.



In this case, the minimum of f on $[d, a]$ occurs at some c with $d < c \leq a$. The same reasoning as before shows that f is decreasing on $(-\infty, c]$ and increasing on $[c, \infty)$. It may also happen that $f(d) < f(a)$ for all $d < a$. Then we may apply the results already proved (for $a < b$) to $d < a$: If the minimum of f ever occurs at a point c in $(d, a)$, then f is decreasing on $(-\infty, c]$ and increasing on $[c, \infty)$, but if the minimum is always at d, then f is increasing on $[d, \infty)$ for all d, so f is increasing.

# CHAPTER 12

1. (ii) $f^{-1}(x) = x^{1/3} + 1$. (If $y = f^{-1}(x)$, then $x = f(y) = (y - 1)^3$, so $y = 1 + x^{1/3}$.)

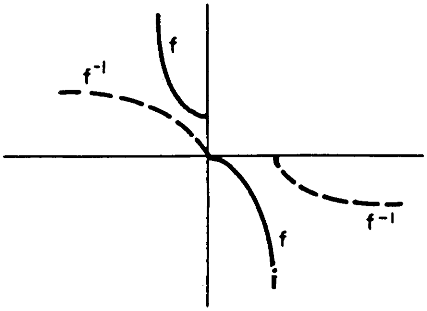
(iv)

$$f^{-1}(x) = \begin{cases} (-x)^{1/2}, & x \leq 0 \\ (1 - x)^{1/3}, & x > 1. \end{cases}$$

(If $y = f^{-1}(x)$, then

$$x = f(y) = \begin{cases} -y^2, & y \geq 0 \\ 1 - y^3, & y < 0. \end{cases}$$

Since $-y^2 \leq 0$ if $y \geq 0$ and $1 - y^3 > 1$ if $y < 0$, we have $y = (-x)^{1/2}$ for $x \leq 0$ and $y = (1 - x)^{1/3}$ for $x > 1$.)



(vi) $f^{-1}(x) = x - [x/2]$ for $[x]$ even. (If $y = f^{-1}(x)$, then

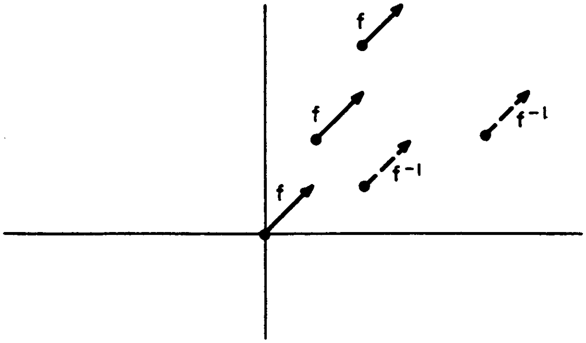
$$\begin{aligned} x = f(y) &= y + [y] \\ &= y + n \qquad \text{for } n \le y < n + 1. \end{aligned}$$

Thus

$$2n \le x < 2n + 1,$$

and

$$y = x - n = x - [x/2].)$$



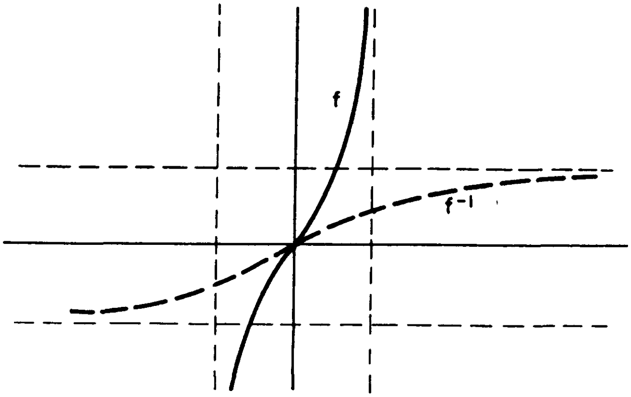
(viii)

$$f^{-1}(x) = \begin{cases} \dfrac{-1 + \sqrt{1 + 4x^2}}{2x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

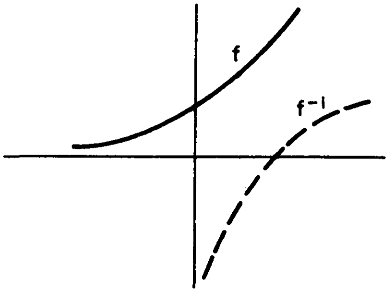
(If $y = f^{-1}(x)$, then $x = f(y) = y/(1 - y^2)$. So $xy^2 + y - x = 0$. If $x = 0$, then $y = 0$. If $x \neq 0$, then

$$y = \frac{-1 + \sqrt{1 + 4x^2}}{2x} \quad \text{or} \quad y = \frac{-1 - \sqrt{1 + 4x^2}}{2x}.$$

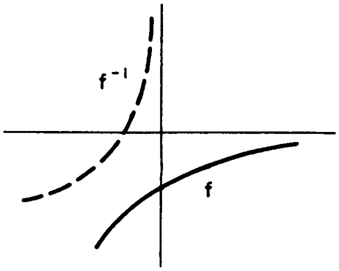
The first possibility is the correct one, since $x$ and $y$ must have the same sign.)



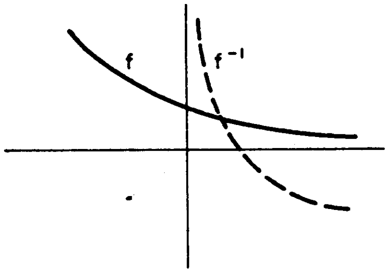
2. (i) $f^{-1}$ is increasing and $f^{-1}(x)$ is not defined for $x \leq 0$.



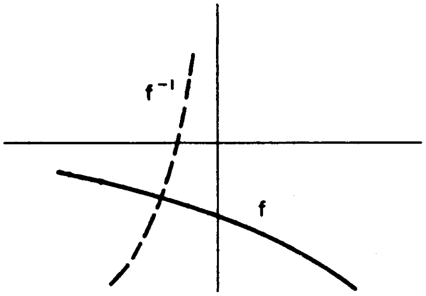
(ii) $f^{-1}$ is increasing and $f^{-1}(x)$ is not defined for $x \geq 0$.



(iii) $f^{-1}$ is decreasing and $f^{-1}(x)$ is not defined for $x \leq 0$.



(iv) $f^{-1}$ is decreasing and $f^{-1}(x)$ is not defined for $x \geq 0$.



6. (b) If $h(x) = 1 + x$, then $g = h \circ f$, so $g^{-1} = f^{-1} \circ h^{-1}$, so $g^{-1}(x) = f^{-1}(x - 1)$. It is also possible to find $g^{-1}$ directly: if $y = g^{-1}(x)$, then $x = g(y)$ $= 1 + f(y)$, so $y = f^{-1}(x - 1)$.)

8. (ii) Any interval $[a, b]$, since f is increasing.

(iv) Those intervals $[a, b]$ which are contained in $(-\infty, 0]$ or $[-1, 1]$ or $[0, \infty)$. (See the graph in the answer to Problem 11-2(v).)

9. Apply Theorem 5 to $f^{-1}$.

10. (a) Let $f = g^{-1}$, where $g(x) = -x^5 - x$. Notice that $g$ is one-one, since $g'(x) = -5x^4 - 1 < 0$, and that $g$ takes on all values. So $f$ is defined on $\mathbf{R}$, and for all $x$ we have

$$x = g(f(x)) = -[f(x)]^5 - f(x).$$

Moreover, $f$ is differentiable, since $g'(x) \neq 0$ for all $x$.

(b)

$$f'(x) = (g^{-1})'(x) = \frac{1}{g'(g^{-1}(x))} \qquad \text{by Theorem 5}$$

$$= \frac{1}{g'(f(x))} = \frac{1}{-5[f(x)]^4 - 1}.$$

(c) Differentiating both sides of

$$[f(x)]^5 + f(x) + x = 0$$

yields

$$5[f(x)]^4 \cdot f'(x) + f'(x) + 1 = 0,$$

so

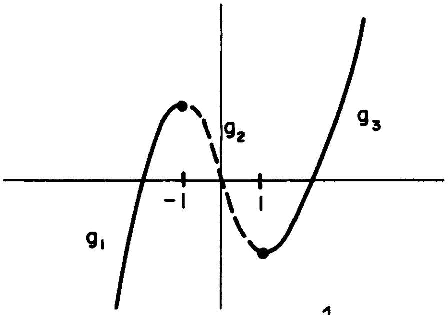
$$f'(x) = \frac{-1}{1 + 5[f(x)]^4}.$$

11. (a) $f(x) = \sqrt{1 - x^2}$ and $f(x) = -\sqrt{1 - x^2}$.

(b) There are no functions with this property.

(c) Let

$$\left.\begin{matrix} g_1(x) \\ g_2(x) \\ g_3(x) \end{matrix}\right\} = g(x) \text{ for } \left\{\begin{matrix} x < -1 \\ -1 < x < 1 \\ x > 1. \end{matrix}\right.$$



Then each $g_i$ is one-one. If $f_i = g_i^{-1}$, then each $f_i$ satisfies $[f_i(x)]^3 - 3f_i(x) = x$. The domain of

$$\left.\begin{matrix} f_1 \\ f_2 \\ f_3 \end{matrix}\right\} \text{ is } \left\{\begin{matrix} (-\infty, 2) \\ (-2, 2) \\ (2, \infty). \end{matrix}\right.$$

(To find $y = f_i(x) = g_i^{-1}(x)$ explicitly we would have to solve the equation $x = g(y) = y^3 - 3y$. This can be done, but only with great difficulty; see Chapter 24.)

It is not hard to see that any continuous function f satisfying $[f(x)]^3 - 3f(x) = x$, and defined on an interval, must be (part of) some $f_i$. For such a function f satisfies $g(f(x)) = x$; this equation implies that f is one-one (Problem 3-23) and that $f^{-1}$ coincides with g on the domain of $f^{-1}$. But the domain of $f^{-1}$ is an interval, and the only intervals on which g is one-one are contained in $(-\infty, -1)$ or $(-1, 1)$ or $(1, \infty)$.

12. (a) Differentiating both sides of $[f(x)]^2 + x^2 = 1$ yields

$$2f(x)f'(x) + 2x = 0,$$

or

$$f'(x) = -\frac{x}{f(x)}.$$

(b) This equation is true for

$$f(x) = \sqrt{1 - x^2}, \text{ in which case } f'(x) = \frac{-x}{\sqrt{1 - x^2}} = \frac{-x}{f(x)},$$

and

$$f(x) = -\sqrt{1 - x^2}, \text{ in which case } f'(x) = \frac{x}{\sqrt{1 - x^2}} = \frac{-x}{f(x)}.$$

(c) We have

$$3[f(x)]^2f'(x) - 3f'(x) = 1,$$

so

$$f'(x) = \frac{1}{3([f(x)]^2 - 1)}.$$

13. Consider a differentiable function f which satisfies

$$[f(x)]^4 + [f(x)]^3 + x\,f(x) = 1;$$

then

$$4[f(x)]^3f'(x) + 3[f(x)]^2f'(x) + f(x) + xf'(x) = 0,$$

$$f'(x) = \frac{-f(x)}{4[f(x)]^3 + 3[f(x)]^2 + x}.$$

16. (ii) $\beta^{-1}(3) = -1$, since $\beta(-1) = h(0) = 3$. So

$$\begin{aligned}(\beta^{-1})'(3) &= \frac{1}{\beta'(\beta^{-1}(3))} = \frac{1}{\beta'(-1)}\\ &= \frac{1}{h'(0)}\\ &= \frac{1}{\sin^2(\sin 1)}.\end{aligned}$$

(The answer is not surprising, since the equation $\beta(x) = h(x + 1)$ implies that $\beta^{-1} = h^{-1} - 1$.)

18. As in Problems 10-16 and 10-28, the main difficulty is in formulating a reasonable conjecture for the form of $(f^{-1})^{(k)}(x)$. It is not hard to prove

the following assertion by induction on k: If $f^{(k)}(f^{-1}(x))$ exists, and is non-zero, then

$$(f^{-1})^{(k)}(x) = \frac{A(x)}{[f'(f^{-1}(x))]^m}$$

for some integer m, where A(x) is a sum of terms of the form

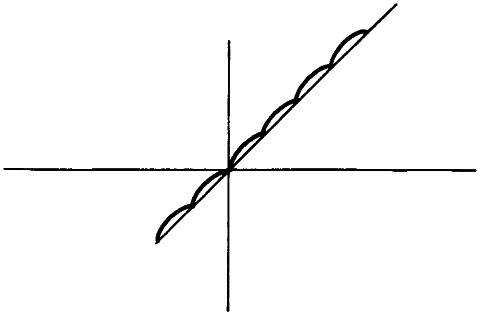
$$[f'(f^{-1}(x))]^{m_1}\cdot \ldots \cdot [f^{(\ell)}(f^{-1}(x))]^{m_\ell}.$$

19. (a) Suppose f is increasing and g is decreasing, and $f(a) = g(a)$. If $a < b$, then

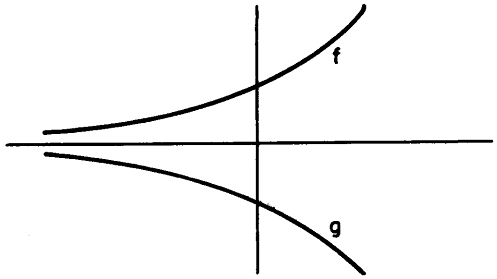
$$g(b) < g(a) = f(a) < f(b),$$

and similarly if $b < a$.

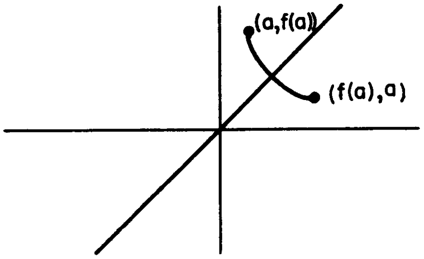
(b) Appropriate functions f and g are shown below (to be explicit we can take $g(x) = x$ and $f(x) = [x] + \sqrt{x - [x]}$ (Problem 4-15)).



(c) Appropriate functions f and g are shown below. (Using the exponential function from Chapter 17, we can define $f(x) = e^x$ and $g(x) = -e^x$, but at the moment explicit definitions would be awkward.)

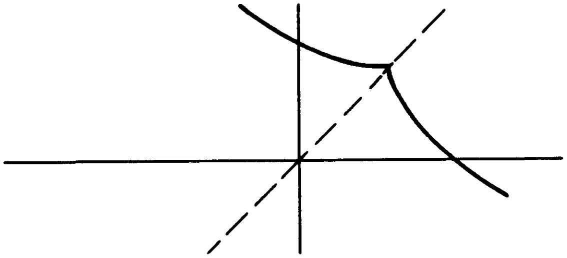


20. (a) The geometric idea behind the proof is indicated below: If $f(a) > a$, then $f(f(a)) = a < f(a)$. Since $f(a) > a$, and $f(b) < b$ for some $b$ (namely, $f(a)$), it follows that $f(x) = x$ for some $x$ in $[a, b]$.



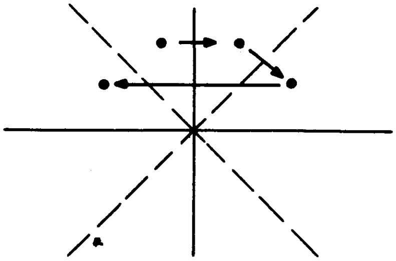
(b) Let $f$ be any decreasing function on $(-\infty, a]$ which takes on all values $\geq a$, and define

$$g(x) = \begin{cases} f(x), & x \leq a \\ f^{-1}(x), & x \geq a. \end{cases}$$



(c) If $f(x) < x$, then $x = f^{-1}(f(x)) < f^{-1}(x) = f(x)$, a contradiction. Similarly, we cannot have $f(x) > x$.

21. The functions with this property are precisely the one-one functions, because reflecting through the antidiagonal is the same as reflecting through the vertical axis, then reflecting through the diagonal, and finally reflecting through the vertical axis again.



If a more analytic proof is desired, notice that the reflection of $(a, b)$ through the antidiagonal is $(-b, -a)$. Thus if $(a, f(a))$ and $(b, f(b))$ are two points in the graph of $f$, we require that $(-f(a), -a)$ and $(-f(b), -b)$ should not have the same first coordinate if $a \neq b$. In other words $f(a)$ and $f(b)$ must be different. So $f$ must be one-one.

22. (a) Since $f$ is not increasing, there is some $x < y$ with $f(x) \leq f(x)$. Since $f$ is nondecreasing, if $x \leq z \leq y$, then $f(x) \leq f(z) \leq f(y) \leq f(x)$. So $f(x) = f(z) = f(y)$.

(b) $f(x + h) \geq f(x)$ for $h > 0$ and $f(x + h) \leq f(x)$ for $h < 0$, so

$$\frac{f(x + h) - f(x)}{h} \geq 0$$

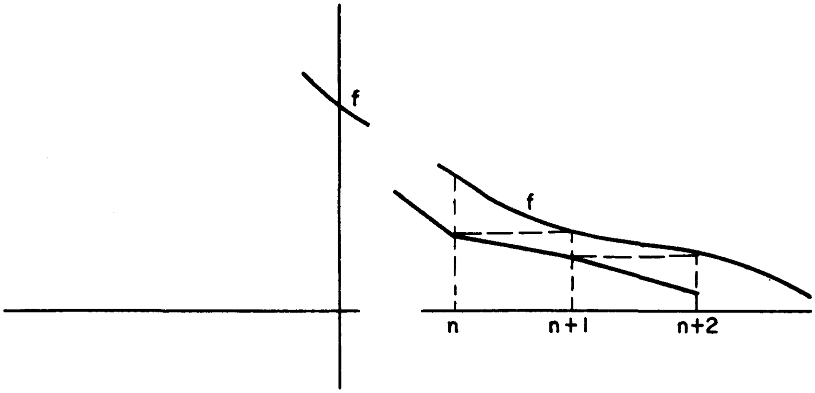
for all $h \neq 0$, so $f'(x) \geq 0$.

(c) If $y > x$, then

$$\frac{f(y) - f(x)}{y - x} = f'(z) \geq 0 \qquad \text{for some } z \text{ in } (x, y),$$

so $f(y) \geq f(x)$. Similarly, if $y < x$, then $f(y) \leq f(x)$.

23. (a) The idea behind the proof is indicated in the figure below. On the interval $[n, n + 1]$, let $g$ be the linear function with $g(n) = f(n + 1)$ and $g(n + 1) = f(n + 2)$.



(b) On the interval $[n, n + 1]$ let $g$ be the linear function with $g(n) = f(n + 1)/(n + 1)$ and $g(n + 2) = f(n + 2)/(n + 2)$.

# CHAPTER 13

3. (a) Problem 2-6 shows that

$$\frac{\sum_{k=1}^{n} k^p}{n^{p+1}} = \frac{1}{p+1} + \frac{A}{n} + \frac{B}{n^2} + \dots,$$

which can clearly be made as close to $1/(p+1)$ as desired by choosing $n$ large enough.

(b) We have

$$L(f, P_n) = \frac{b^{p+1}}{n^{p+1}} \left[ \sum_{k=0}^{n-1} k^p \right],$$

$$U(f, P_n) = \frac{b^{p+1}}{n^{p+1}} \left[ \sum_{k=1}^{n} k^p \right].$$

Part (a) shows that $L(f, P_n)$ and $U(f, P_n)$ can be made as close to $b^{p+1}/(p+1)$ as desired by choosing $n$ sufficiently large. As in Problem 1, this implies that $\int_0^b x^p\,dx = b^{p+1}/(p+1)$.

4. (ii) $\int_0^2 f = 0.$

(iv) $f$ is not integrable.

(vi) $f$ is integrable; a rigorous proof can be given in several ways, using various problems in this chapter, for example Problem 15. (Presumably, the integral of $f$ is

$$\frac{1}{2}+\frac{1}{2}(\frac{1}{2}-\frac{1}{3})+\frac{1}{3}(\frac{1}{3}-\frac{1}{4})+\cdots .$$

At the moment we do not even know that an infinite sum means, let alone how to work with them, but the following likely looking manipulations are actually valid:

$$\frac{1}{2}+\frac{1}{2}(\frac{1}{2}-\frac{1}{3})+\frac{1}{3}(\frac{1}{3}-\frac{1}{4})+\cdots$$

$$=(\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots)+\frac{1}{2}-\frac{1}{2\cdot 3}-\frac{1}{3\cdot 4}-\cdots$$

$$=(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots)-\frac{1}{2}-\frac{1}{2\cdot 3}-\frac{1}{3\cdot 4}-\cdots$$

$$=(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots)-(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots).$$

From the fact that

$$\frac{1}{1\cdot 2}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1},$$

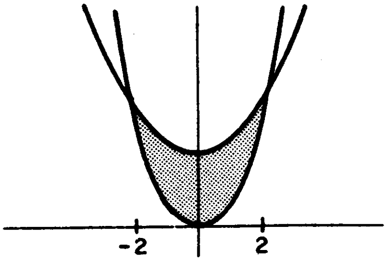
derived in Problem 2-5, we might guess that

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots=1.$$

The other infinite sum happens to equal $\pi^2/6$ (but we will not get to a proof of this fact anywhere in the text), so the integral of f is $\pi^2/6 - 1$.)

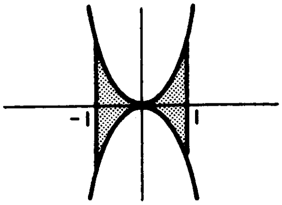
5. (i)

$$\int_{-2}^{2} \left(\frac{x^2}{2} + 2\right) - x^2 \, dx = \frac{16}{3}.$$



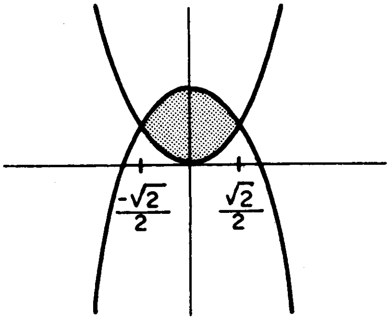
(ii)

$$\int_{-1}^{1} x^2 - (-x^2) \, dx = \frac{4}{3}.$$



(iii)

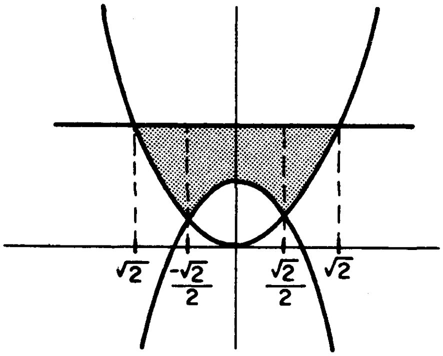
$$\int_{-\sqrt{2}/2}^{\sqrt{2}/2} (1 - x^2) - x^2 \, dx = \frac{2\sqrt{2}}{3}.$$



(iv)

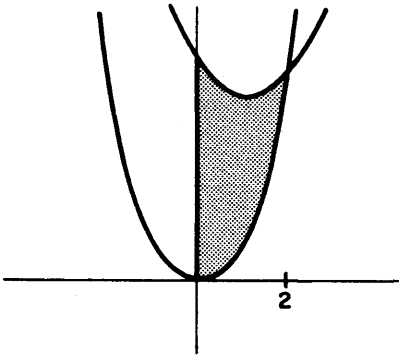
$$\int_{-\sqrt{2}}^{-\sqrt{2}/2} 2 - x^2 \, dx + \int_{-\sqrt{2}/2}^{\sqrt{2}/2} 2 - (1 - x^2) \, dx + \int_{\sqrt{2}/2}^{\sqrt{2}} 2 - x^2 \, dx$$

$$= 2\sqrt{2}.$$



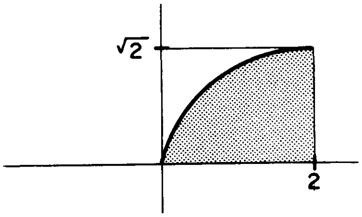
(v)

$$\int_0^2 (x^2 - 2x + 4) - x^2 \, dx = 4.$$



(vi) The area should be

$$2\sqrt{2} - \int_0^{\sqrt{2}} x^2\,dx = \frac{4\sqrt{2}}{3}.$$



7. The first inequality is a special case of Problem 8-13, and the second inequality follows from the fact that $\{f(x_1) + g(x_2): t_{i-1} \le x_1, x_2 \le t_i\}$ contains all numbers in $\{f(x) + g(x): t_{i-1} \le x \le t_i\}$, and possibly some smaller ones.

8. (a) If $L(f, P) = U(f, P)$ for even one partition $P$, then each $m_i = M_i$, so $f$ is constant on each $[t_{i-1}, t_i]$. Since these closed intervals overlap, $f$ must be constant on all of $[a, b]$.

   (b) If $L(f, P_1) = U(f, P_2)$ and $P$ contains both $P_1$ and $P_2$, then $L(f, P_1) \le L(f, P) \le U(f, P) \le U(f, P_2) = L(f, P_1)$, so $L(f, P) = U(f, P)$. It follows from part (a) that $f$ is constant on $[a, b]$.

   (c) Only constant functions. For suppose $f$ is not constant on $[a, b]$, and let $m$ be the minimum value of $f$ on $[a, b]$. Since $f(x) > m$ for some $x$, and since $f$ is continuous, we can choose a partition $P = \{t_0, \ldots, t_n\}$ of $[a, b]$ so that $f > m$ on some interval $[t_{i-1}, t_i]$. Then $m_i > m$, so

$L(f, P) > m(b - a)$. On the other hand, if $Q$ is the partition $Q = \{a, b\}$, then $L(f, Q) = m(b - a)$.

(d) If $f$ is integrable on $[a, b]$ and all lower sums are equal, then $f$ takes on the value $m = \inf\{f(x)\colon a \le x \le b\}$ at a dense set of points in $[a, b]$. In fact, Problem 21 shows that $f$ is continuous at a dense set of points. Now if $f$ is continuous at $x$ and $f(x) > m$, then, as in part (c), there is a partition $P$ with $L(f, P) > m(b - a)$, while $L(f, Q) = m(b - a)$ if $Q = \{a, b\}$, contradicting the hypothesis. Conversely, it is easy to see that if $f$ takes on its minimum value $m$ on a dense set of points in $[a, b]$, then $L(f, P) = m(b - a)$, since each $m_i = m$. (The condition that $f$ be integrable is essential in this problem. For example, if $f(x) = 1/q$ for $x = p/q$ in lowest terms, and $f(x) = 1$ for $x$ irrational, then $L(f, P) = 0$ for all $P$, but $f$ does not take on the value $0 = \inf\{f(x)\colon a \le x \le b\}$ anywhere.)

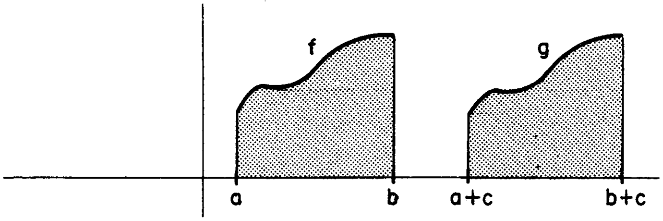
9. Theorem 4, applied to $a < b < d$, implies that $f$ is integrable on $[b, d]$. Then Theorem 4, applied to $b < c < d$, implies that $f$ is integrable on $[b, c]$.

11. Let $P = \{t_0, \dots, t_n\}$ be a partition of $[a, b]$. If $g(x) = f(x - c)$, then

$$m_i = \inf\{f(x)\colon t_{i-1} \le x \le t_i\} = \inf\{g(x)\colon t_{i-1} + c \le x \le t_i + c\}$$

and similarly for $M_i$, so $L(f, P) = L(g, P')$ and $U(f, P) = U(g, P')$. If f is integrable, so that for every $\epsilon > 0$ we have $U(f, P) - L(f, P) < \epsilon$ for some P, then g is also integrable, since we have $U(g, P') - L(g, P') < \epsilon$. Moreover,

$$\int_a^b f(x)\,dx = \sup\{L(f, P)\} = \sup\{L(g, P')\} = \int_{a+c}^{b+c} f(x - c)\,dx.$$



12. Notice that

$$b\cdot\inf\{1/t\colon t_{i-1} \le t \le t_i\} = \inf\{1/t\colon bt_{i-1} \le x \le bt_i\}.$$

Denoting the first inf by $m_i$ and the second by $m_i'$, we have

$$\begin{aligned} L(f, P') &= \sum_{i=1}^{n} m_i'(bt_i - bt_{i-1}) \\ &= \sum_{i=1}^{n} bm_i'(t_i - t_{i-1}) \\ &= \sum_{i=1}^{n} m_i(t_i - t_{i-1}) \\ &= L(f, P). \end{aligned}$$

So

$$\int_b^{ab} 1/t\,dt = \sup\{L(f, P')\} = \sup\{L(f, P)\} = \int_1^a 1/t\,dt.$$

13. If $P = \{t_0,\dots,t_n\}$ is a partition of $[a, b]$, and $P' = \{ct_0,\dots,ct_n\}$, then

$$m_i = \inf\{f(ct) : t_{i-1} \le t \le t_i\} = \inf\{f(t) : ct_{i-1} \le t \le ct_i\} = m_i'.$$

So if $g(t) = f(ct)$, then

$$\begin{aligned} cL(g, P') &= c\sum_{i=1}^{n} m_i(t_i - t_{i-1}) \\ &= \sum_{i=1}^{n} m_i(ct_i - ct_{i-1}) \\ &= L(f, P'). \end{aligned}$$

So

$$\begin{aligned} \int_{ca}^{cb} f(t)\,dt = \sup\{L(f, P')\} &= c\cdot\sup\{L(g, P)\} \\ &= c\cdot\int_a^b f(ct)\,dt. \end{aligned}$$

(Actually, this proof is valid only for $c \ge 0$, but the case $c < 0$ can then be deduced easily.)

14. Choose $M \ge 1$ so that $|f(x)| \le M$ for all $x$ in $[a, b]$. Given $\epsilon > 0$, let $\delta = \epsilon/3M$. Since $f$ is continuous on $[a, x_0 - \delta/2]$ and $[x_0 + \delta/2, b]$ there are

partitions $P_1 = \{t_0, \dots, t_n\}$ of $[a, x_0 - \delta/2]$ and $P_2 = \{s_0, \dots, s_m\}$ of $[x_0 + \delta/2, b]$ such that $U(f, P_1) - L(f, P_1) < \epsilon/3$ and $U(f, P_2) - L(f, P_2) < \epsilon/3$. If $P = \{t_0, \dots, t_n, s_0, \dots, s_m\}$, then

$$\begin{aligned} U(f, P) - L(f, P) &\leq [U(f, P_1) - L(f, P_1)] + \delta \cdot M + [U(f, P_2) - L(f, P_2)] \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \end{aligned}$$

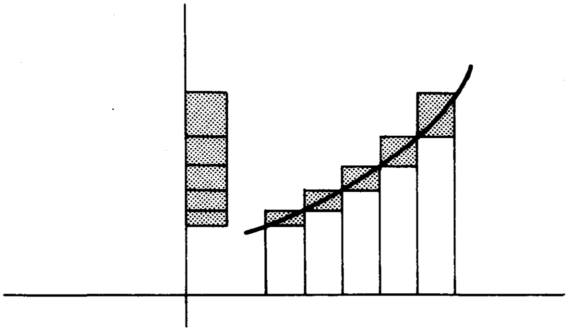
5. (a)

$$L(f, P) = \sum_{i=1}^{n} f(t_{i-1})(t_i - t_{i-1}),$$

$$U(f, P) = \sum_{i=1}^{n} f(t_i)(t_i - t_{i-1}).$$

(b) If $t_i - t_{i-1} = \delta$ for each $i$, then

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^{n} [f(t_i) - f(t_{i-1})](t_i - t_{i-1}) \\ &= \delta \sum_{i=1}^{n} f(t_i) - f(t_{i-1}) \\ &= \delta[f(b) - f(a)]. \end{aligned}$$



(c) For every $\epsilon > 0$ we have $U(f, P) - L(f, P) < \epsilon$ if $t_i - t_{i-1} = \delta = \epsilon/[f(b) - f(a)]$.

(d) The function in Problem 4(vi) is an example (on the interval $[0, 1]$).

16. (a)

$$\begin{aligned} L(f^{-1}, P) + U(f, P') &= \sum_{i=1}^{n} f^{-1}(t_{i-1})(t_i - t_{i-1}) + \sum_{i=1}^{n} t_i(f^{-1}(t_i) - f^{-1}(t_{i-1})) \\ &= \sum_{i=1}^{n} [t_i f^{-1}(t_i) - t_{i-1}f^{-1}(t_{i-1})] \\ &= bf^{-1}(b) - af^{-1}(a). \end{aligned}$$

(b) It follows from part (a) that

$$\begin{aligned} \int_a^b f^{-1} = \sup\{L(f^{-1}, P)\} &= \sup\{bf^{-1}(b) - af^{-1}(a) - U(f, P')\} \\ &= bf^{-1}(b) - af^{-1}(a) - \inf\{U(f, P')\} \\ &= bf^{-1}(b) - af^{-1}(a) - \int_{f^{-1}(a)}^{f^{-1}(b)} f. \end{aligned}$$

(c) If $f(x) = x^n$ for $x \geq 0$, then for $0 \leq a < b$ we have

$$\int_a^b \sqrt[n]{x}\, dx = \int_a^b f^{-1} = bf^{-1}(b) - af^{-1}(a) - \int_{f^{-1}(a)}^{f^{-1}(b)} x^n\, dx$$

$$= b\sqrt[n]{b} - a\sqrt[n]{a} - \left[\frac{(\sqrt[n]{b})^{n+1}}{n+1} - \frac{(\sqrt[n]{a})^{n+1}}{n+1}\right]$$

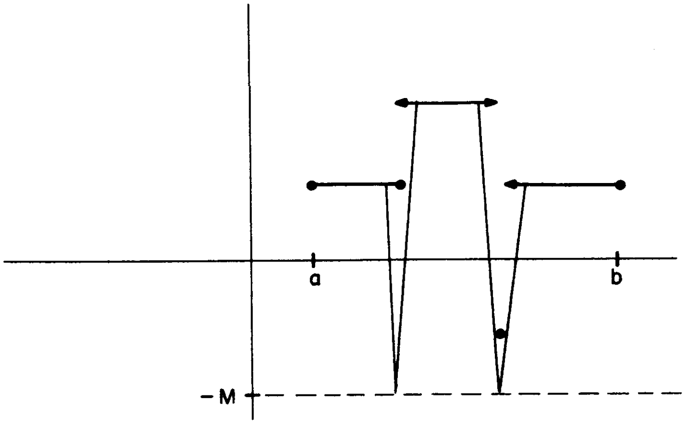
$$= \frac{n\sqrt[n]{b}}{n+1} - \frac{n\sqrt[n]{a}}{n+1}.$$

17. (a) If $P = \{t_0,\dots,t_n\}$ is a partition of $[a, b]$ with $U(f, P) - L(f, P) < \epsilon$, then $U(f, P) - \int_a^b f < \epsilon$ and $\int_a^b f - L(f, P) < \epsilon$. Let $s_1(x)$ be $m_i$ for $x$ in $(t_{i-1}, t_i)$ and 0, say, for $x = t_0,\dots,t_n$; similarly let $s_2(x)$ be $M_i$ for $x$ in $(t_{i-1}, t_i)$ and 0 for $x = t_0,\dots,t_n$.

(b) The existence of such step functions implies the existence of partitions $P_1$ and $P_2$ with $U(f, P_2) - L(f, P_1) < \epsilon$.

(c) The function in Problem 25 is an example.

18. Choose a step function $s \le f$ with $\int_a^b f - \int_a^b s < \epsilon/2$, by Problem 17(a). Choose $M \ge 1$ so that $|f(x)| \le M$ for all $x$ in $[a, b]$, and if $s$ is constant on $(t_{i-1}, t_i)$ for $i = 1,\dots,n$, choose $\delta < \epsilon/2nM$. Let $g = s$ on $[t_{i-1} + \delta/2, t_i - \delta/2]$ and let $g$ be a linear function on $[t_i - \delta/2, t_i]$ and $[t_i, t_i + \delta/2]$ with $g(t_i) = -M$.



Then $g \leq s \leq f$ and $\int_a^b s - \int_a^b g \leq nM\delta < \epsilon/2$, so $\int_a^b f - \int_a^b g < \epsilon$.

19. (a) If $s_1$ (respectively $s_2$) is constant on each subinterval for a partition $P_1$ (respectively $P_2$), then $s_1 + s_2$ is constant on the intervals for the partition P which contains $P_1$ and $P_2$.

(b) Part (a) shows that there is a partition $P = \{t_0,\ldots,t_n\}$ such that $s_1$ and $s_2$ are constant on each $(t_{i-1}, t_i)$, with values $a_i$ and $b_i$, say.

Then

$$\begin{aligned}\int_a^b (s_1 + s_2) &= \sum_{i=1}^{n} (a_i + b_i)(t_i - t_{i-1}) \\ &= \sum_{i=1}^{n} a_i(t_i - t_{i-1}) + \sum_{i=1}^{n} b_i(t_i - t_{i-1}) \\ &= \int_a^b s_1 + \int_a^b s_2.\end{aligned}$$

(c) Given $\epsilon > 0$, choose step functions $s_1$, $s_2$ and $t_1$, $t_2$ with $s_1 \le f \le s_2$ and $t_1 \le g \le t_2$ and $\int_a^b s_2 - \int_a^b s_1 < \epsilon/2$ and $\int_a^b t_2 - \int_a^b t_1 < \epsilon/2$. Part (b) implies that

$$\int_a^b (s_1 + t_1) = \int_a^b s_1 + \int_a^b t_1 \le \int_a^b f + \int_a^b g \le \int_a^b s_2 + \int_a^b t_2 = \int_a^b s_2 + t_2$$

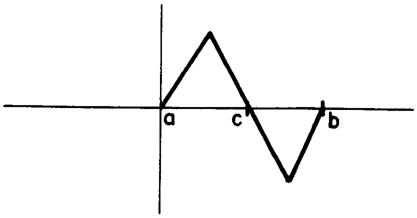
and that

$$\int_a^b (s_2 + t_2) - \int_a^b (s_1 + t_1) < \epsilon.$$

This shows that f is integrable, and also that $\int_a^b (f + g) = \int_a^b f + \int_a^b g$, since there is only one number between all such $\int_a^b (s_2 + t_2)$.

20. Let $g(x) = \int_a^x f - \int_x^b f$. Then g is continuous and $g(a) = -\int_a^b f$ and $g(b) = \int_a^b f$; so $g(a)$ and $g(b)$ have different signs and consequently $g(x) = 0$ for some x in $[a, b]$, unless $g(a) = 0$, in which case we can choose $x = a$.

For the function f showing below, only $x = a$ or $x = b$ will work; f has been chosen so that $\int_a^c f = -\int_c^b f$.



21. (a) Clearly if $M_i - m_i \geq 1$ for all i, then $U(f, P) - L(f, P) \geq b - a$.

(b) If $i = 1$, let $b_1 = t_1$ and choose any $a_1$ with $t_0 < a_1 < t_1$. Similarly if $i = n$.

(c) Choose a partition P of $[a_1, b_1]$ with $U(f, P) - L(f, P) < (b_1 - a_1)/2$. Then $M_i - m_i < 1/2$ for some i. Choose $[a_2, b_2] = [t_{i-1}, t_i]$ unless $i = 1$ or $n$, in which case use the modification of part (b).

(d) Let x be a point in each $I_n$. Notice that we cannot have $x = a_n$ or $b_n$, since x is also in $[a_{n+1}, b_{n+1}]$ and $a_n < a_{n+1} < b_{n+1} < b_n$. If $\epsilon > 0$, there is some n such that $\sup\{f(x)\colon x \text{ in } I_n\} - \inf\{f(x)\colon x \text{ in } I_n\} < \epsilon/2$. Then $|f(y) - f(x)| < \epsilon$ for all y in $I_n$; since x is in $(a_n, b_n)$, this means that $|f(y) - f(x)| < \epsilon$ for all y satisfying $|y - x| < \delta$ where $\delta > 0$ is the minimum of $x - a_n$ and $b_n - x$. Thus f is continuous at x.

(e) f must be continuous at some point in every interval contained in [a, b], since f is integrable on every such interval.

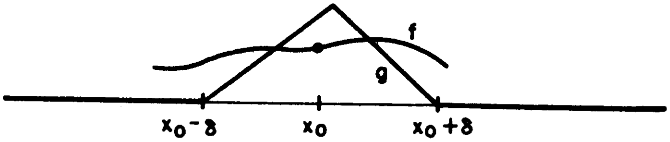
22. (a) Choose $x_0$ in [a, b] and let $f(x) = 0$ for all $x \neq x_0$, and $f(x_0) = 1$. (The function in Problem 25 is another example.)

(b) There is a partition P of [a, b] such that $f(x) > x_0/2$ for all x in some $[t_{i-1}, t_i]$. Then $L(f, P) \geq x_0(t_i - t_{i-1})/2$.

(c) This follows from part (b), since $f$ is continuous at some $x_0$, by Problem 21.

23. (a) Choose $g = f$. Then $\int_a^b f^2 = 0$. Since $f$ is continuous, this implies that $f = 0$.

(b) If $f(x_0) > 0$, then $f(x) > 0$ for all $x$ in $(x_0 - \delta, x_0 + \delta)$ for some $\delta > 0$. Choose a continuous $g$ with $g > 0$ on $(x_0 - \delta, x_0 + \delta)$ and $g = 0$ elsewhere. Then $\int_a^b fg > 0$, a contradiction.



25. Let $\epsilon > 0$. Choose $n$ so that $1/n < \epsilon/2$. Let $x_0 < x_1 < \dots < x_m$ be those rational points $p/q$ in $[0, 1]$ with $q < n$. Choose a partition $P = \{t_0,\dots,t_k\}$ such that the intervals $[t_{i-1}, t_i]$ which contain some $x_j$ have total length $< \epsilon/2$. On each of the other intervals we have $M_i \le 1/n < \epsilon/2$. Let $I_1$ denote all those $i$ from $1,\dots,n$ for which $[t_{i-1}, t_i]$ contains some $x_j$, and let $I_2$ denote all other $i$ from $1,\dots,n$. Since $f \le 1$ everywhere, we have

$$
\begin{aligned}
U(f, P) &= \sum_{i \text{ in } I_1} M_i(t_i - t_{i-1}) + \sum_{i \text{ in } I_2} M_i(t_i - t_{i-1}) \\
&\le 1\cdot \sum_{i \text{ in } I_1} (t_i - t_{i-1}) + \frac{\epsilon}{2} \sum_{i \text{ in } I_2} (t_i - t_{i-1}) \\
&\le 1\cdot\frac{\epsilon}{2} + \frac{\epsilon}{2}\cdot 1 = \epsilon.
\end{aligned}
$$

26. Let f be the function in Problem 25, and let $g(x) = 0$ for $x = 0$, and $g(x) = 1$ for $x \neq 0$. Then $(g \circ f)(x) = 0$ if x is irrational, and 1 if x is rational.

28. (a) From the inequality $mg(x) \leq f(x)g(x) \leq Mg(x)$, we obtain

$$m \int_a^b g(x)\ dx \leq \int_a^b f(x)g(x)\ dx \leq M \int_a^b g(x)\ dx.$$

Consequently

$$\int_a^b f(x)g(x)\ dx = \mu \int_a^b g(x)\ dx$$

for some $\mu$ with $m \leq \mu \leq M$. This $\mu = f(\xi)$ for some $\xi$ in $[a, b]$.

(b) If $g(x) = x$ on $[-1, 1]$ and $f(x) = x$, then

$$\int_{-1}^{1} f(x)g(x)\ dx = \int_{-1}^{1} x^2\ dx = \frac{2}{3} \neq \mu \cdot \int_{-1}^{1} x\ dx.$$

29. (a) Let

$$h(\xi) = g(a) \int_a^\xi f(t)\ dt + g(b) \int_\xi^b f(t)\ dt.$$

Then

$$h(a) = g(b) \int_a^b f(t)\ dt,$$
$$h(b) = g(a) \int_a^b f(t)\ dt.$$

Since h is continuous, h takes on all values between $g(a)\int_a^b f(t)\,dt$ and $g(b)\int_a^b f(t)\,dt$. On the other hand,

$$g(a)f(t) \le g(t)f(t) \le g(b)f(t) \qquad \text{for all } t \text{ in } [a, b],$$

provided that $g(t) \ge 0$ for all t in [a, b]. So in this case

$$g(a)\int_a^b f(t)\,dt \le \int_a^b f(t)g(t)\,dt \le g(b)\int_a^b f(t)\,dt$$

and consequently $\int_a^b f(t)g(t)\,dt = h(\xi)$ for some $\xi$.

The case where g(t) is not $\ge 0$ for all t in [a, b] can now be taken care of by a simple trick: replace g by g + c where $g(t) + c \ge 0$ for all t in [a, b]. The theorem for g + c implies the theorem for g.

(b) Let f = 1 and let g be any function on [a, b] with $g(a) = g(b) = 0$ but $g(x) > 0$ for all x in (a, b).

30. (a) If

$$\left.\begin{array}{l} 0 \le m_i \le M_i, \\ m_i \le M_i \le 0, \\ m_i \le 0 \le M_i, \end{array}\right\} \text{ then } \left\{\begin{array}{l} M_i' = M_i = \max(|M_i|, |m_i|) \\ M_i' = -m_i = \max(|M_i|, |m_i|) \\ M_i' = \max(M_i, -m_i) = \max(|M_i|, |m_i|). \end{array}\right.$$

The proof for $m_i'$ is similar.

(b) If $|m_i| \leq |M_i|$, then

$$M_i' - m_i' = |M_i| - |m_i| \leq |M_i - m_i| = M_i - m_i;$$

if $|M_i| \leq |m_i|$, then

$$M_i' - m_i' = |m_i| - |M_i| \leq |m_i - M_i| = M_i - m_i.$$

(c) If P is a partition of [a, b], then

$$\begin{aligned} U(|f|, P) - L(|f|, P) &= \sum_{i=1}^{n} (M_i' - m_i')(t_i - t_{i-1}) \\ &\leq \sum_{i=1}^{n} (M_i - m_i)(t_i - t_{i-1}) \\ &= U(f, P) - L(f, P). \end{aligned}$$

So integrability of f implies integrability of $|f|$, by Theorem 2.

(d) This follows from part (c) and the formulas $\max(f, g) = [f + g + |f - g|]/2$, $\min(f, g) = [f + g - |f - g|]/2$.

(e) If f is integrable, then $\max(f, 0)$ and $\min(f, 0)$ are integrable, by part (d). Conversely, if $\max(f, 0)$ and $\min(f, 0)$ are integrable, then $f = \max(f, 0) + \min(f, 0)$ is integrable, by Theorem 5.

32. (a) Since

$$0 \leq m_i' \leq f(x) \leq M_i' \quad \text{and} \quad m_i'' \leq g(x) \leq M_i'' \qquad \text{for all } x \text{ in } [t_{i-1}, t_i],$$

we have

$$m_i'm_i'' \leq f(x)g(x) \leq M_i'M_i'' \qquad \text{for all } x \text{ in } [t_{i-1}, t_i],$$

which implies that $m_i'm_i'' \leq m_i$ and $M_i \leq M_i'M_i''$.

(b) This follows immediately from part (a).

(c) By part (b),

$$\begin{aligned} U(fg, P) - L(fg, P) &\leq \sum_{i=1}^{n} [M_i'M_i'' - m_i'm_i''](t_i - t_{i-1}) \\ &= \sum_{i=1}^{n} M_i''[M_i' - m_i'](t_i - t_{i-1}) + \sum_{i=1}^{n} m_i'[M_i'' - m_i''](t_i - t_{i-1}) \\ &\leq M\left\{\sum_{i=1}^{n} [M_i' - m_i'](t_i - t_{i-1}) + \sum_{i=1}^{n} [M_i'' - m_i''](t_i - t_{i-1})\right\}. \end{aligned}$$

(d) Integrability of fg follows immediately from part (c) and Theorem 2.

(e) $f \cdot g = g \cdot \max(f, 0) + g \cdot (-\min(f, 0))$, both of which are integrable by part (d).

33. (a) The following proof is modelled on Problem 1-18(b). If $g = 0$, then equality holds. Otherwise we have

$$0 \leq \int_a^b (f - \lambda g)^2 = \int_a^b f^2 - 2\lambda \int_a^b fg + \lambda^2 \int_a^b g^2,$$

so

$$\int_a^b f^2 - \frac{4(\int_a^b fg)^2}{4(\int_a^b g^2)} \geq 0.$$

(It is not possible to obtain strict inequality when $f \neq \lambda g$, since $\int_a^b (f - \lambda g)$ might be 0 even if $f \neq \lambda g$; see part (b).)

(b) If $f = g$ except at one point, then equality holds, even though $f = \lambda g$ is false. But if f and g are continuous, and $g \neq 0$, then $f = \lambda g$ for some $\lambda$, for if $f \neq \lambda g$ for all $\lambda$, then

$$0 < \int_a^b (f - \lambda g)^2 = \int_a^b f^2 - 2\lambda \int_a^b fg + \lambda^2 \int_a^b g^2,$$

since $(f - \lambda g)^2$ is a continuous non-negative function which is somewhere positive. This implies strict inequality in the Cauchy-Schwartz inequality.

(c) Given $x_1,\dots,x_n$ and $y_1,\dots,y_n$, let f and g be defined on [0, 1] by

$$f(x) = \begin{cases} x_i, & \frac{i-1}{n} \leq x < \frac{i}{n} \\ 0, & x = 1, \end{cases}$$

$$g(x) = \begin{cases} y_i, & \frac{i-1}{n} \leq x < \frac{i}{n} \\ 0, & x = 1. \end{cases}$$

Then

$$\int_0^1 fg = \frac{1}{n^2}\sum_{i=1}^{n} x_i y_i,$$

$$\int_0^1 f^2 = \frac{1}{n^2}\sum_{i=1}^{n} x_i{}^2,$$

$$\int_0^1 g^2 = \frac{1}{n^2}\sum_{i=1}^{n} y_i{}^2,$$

so

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \le \left(\sum_{i=1}^{n} x_i{}^2\right)\left(\sum_{i=1}^{n} y_i{}^2\right);$$

this is the Schwartz inequality. (Problem 1-18 is actually the special case $n = 2$, but the proofs in that problem will work for all $n$.)

(d) Apply the Cauchy-Schwartz inequality to $f$ and $g(x) = 1$ on $[0, 1]$. The correct result for $[a, b]$ is

$$\left(\int_a^b f\right)^2 \le (b - a)\left(\int_a^b f^2\right).$$

34. If $\epsilon > 0$, pick $N \ge 0$ so that $|f(t) - a| < \epsilon$ for $t \ge N$. Then for $N \ge 0$ we have

$$\left|\int_N^{N+M} f(t)\,dt - Ma\right| < \epsilon M,$$

so

$$\left|\frac{1}{N+M}\int_N^{N+M} f(t)\,dt - \frac{Ma}{N+M}\right| < \frac{\epsilon M}{N+M} < \epsilon.$$

Choose M so that

$$\left|\frac{Ma}{N+M} - a\right| < \epsilon \qquad \text{and} \qquad \left|\frac{1}{N+M}\int_1^N f(t)\,dt\right| < \epsilon.$$

Then

$$\left|\frac{1}{N+M}\int_1^{N+M} f(t)\,dt - a\right| < 3\epsilon.$$

# CHAPTER 14

1. (ii) $$\frac{1}{1+\sin^6\left(\int_1^x \sin^3 t\,dt\right)+\left(\int_1^x \sin^3 t\,dt\right)^2}\cdot\sin^3 x.$$

(iv) $$\frac{-1}{1+x^2+\sin^2 x}.$$

(vi) $$\cos\left(\int_0^x \sin\left(\int_0^y \sin^3 t\,dt\right)dy\right)\cdot\sin\left(\int_0^x \sin^3 t\,dt\right).$$

(viii) $$(F^{-1})'(x)=\frac{1}{F'(F^{-1}(x))}=\frac{1}{\dfrac{1}{\sqrt{1-[F^{-1}(x)]^2}}}=\sqrt{1-[F^{-1}(x)]^2}.$$

2. (ii) All $x \neq 1$.

(iv) All irrational $x$.

(vi), (viii) All $x$ not of the form $1/n$ for some natural number $n$.

3. (ii)

$$(f^{-1})'(0)=\frac{1}{f'(f^{-1}(0))}=\frac{1}{\sin(\sin(f^{-1}(0)))}$$

$$=\frac{1}{\sin(\sin(1))}.$$

5. If $F(x) = \int_0^x f(u)(x - u)\,du = \int_0^x xf(u)\,du - \int_0^x uf(u)\,du$, then

$$F'(x) = [xf(x) + \int_0^x f(u)\,du] - xf(x) \qquad \text{by Problem 4}$$

$$= \int_0^x f(u)\,du.$$

Consequently, there is some number c such that

$$\int_0^x f(u)(x - u)\,du = \int_0^x \left(\int_0^u f(t)\,dt\right) du + c \qquad \text{for all } x.$$

Clearly $c = 0$, since each of the other two terms is 0 for $x = 0$.

6. Applying Problem 5 to $g(u) = f(u)(x - u)$, we obtain

$$\int_0^x f(u)(x - u)^2\,du = \int_0^x [f(u)(x - u)](x - u)\,du$$

$$= \int_0^x \left(\int_0^u f(t)(x - t)\,dt\right) du.$$

Therefore we must show that

$$\int_0^x \left(\int_0^u f(t)(x - t)\,dt\right) du = 2\int_0^x \left(\int_0^{u_2} \left(\int_0^{u_1} f(t)\,dt\right) du_1\right) du_2.$$

Now $x - t = (u - t) + (x - u)$, so

$$(1)\qquad \int_0^u f(t)(x-t)\,dt = \int_0^u f(t)(u-t)\,dt + \int_0^u f(t)(x-u)\,dt.$$

For the first integral on the right we have

$$(2)\qquad \int_0^u f(t)(u-t)\,dt = \int_0^u\left(\int_0^{u_1} f(t)\,dt\right)du_1$$

by Problem 5. The second can be written

$$(3)\qquad \int_0^u f(t)(x-u)\,dt = (x-u)\int_0^u f(t)\,dt.$$

From (1), (2), and (3) we have

$$\int_0^x\left(\int_0^u f(t)(x-t)\,dt\right)du = \int_0^x\left(\int_0^u\left(\int_0^{u_1} f(t)\,dt\right)du_1\right)du$$

$$+\int_0^x \left[(x-u)\int_0^u f(t)\,dt\right]du.$$

On the other hand, applying Problem 5 to $g(u) = \int_0^u f(t)\,dt$ we obtain

$$\int_0^x \left[(x-u)\int_0^u f(t)\,dt\right]du = \int_0^x\left(\int_0^u\left(\int_0^{u_1} f(t)\,dt\right)du_1\right)du.$$

8. (a) This follows from Problem 13-11, since $f(x-a) = f(x)$ for all $x$.

(b) Let $g$ be periodic and continuous with $g \geq 0$ (for example, $g(x) = \sin^2 x$). If $f(x) = \int_0^x g$, then $f' = g$ is periodic, but $f$ is increasing, so it is not periodic.

(c) Let $g(x) = f(x + a)$. Then $g'(x) = f'(x + a) = f'(x)$. If $f(a) = f(0)$, then we also have $g(0) = f(a) = f(0)$. Consequently $g = f$, i.e., $f(x + a) = f(x)$ for all $x$.

On the other hand, if $f$ is periodic with period $a$, then surely $f(a) = f(0)$. Actually, the Problem, as stated, is more difficult, since it was not stipulated that the period of $f$ should be $a$. In other words, it must still be shown that if $f$ is periodic with some period $b \neq a$, then $f(a) = f(0)$ (and consequently $f$ is also periodic with period $a$). This can be done as follows. Suppose that $f(a) \neq f(0)$ and, to be explicit, say that $f(a) > f(0)$. Since the function $g(x) = f(x + a) - f(x)$ satisfies $g'(x) = f'(x + a) - f'(x) = 0$, it follows that $g(x) = g(0) = f(a)$, so $f(x + a) = f(x) + f(a)$ for all $x$. Consequently $f(na) = n \cdot f(a) > f(0)$ for all natural numbers $n$. Moreover, a stronger statement can be made. Since $f$ is continuous and $f(a) > f(0)$, there is some $\epsilon > 0$ such that $f(x) > f(0)$ for $|x - a| < \epsilon$; the relation $f(x + a) = f(x) + f(a)$ implies that we also have $f(x) > f(0)$ for $|x - na| < \epsilon$. If we assume that $f$ is periodic with period $b$, then $f(mb) = f(0)$ for all natural numbers $m$. This will lead to a contradiction, if we can show that for some $m$ and $n$ we have $|mb - na| < \epsilon$, or equivalently,

$$\left|\frac{m}{n} - \frac{a}{b}\right| < \frac{\epsilon}{an}.$$

If $a/b$ is rational, this is clear. If $a/b$ is irrational, then the desired inequality follows from the fact that for any irrational $\alpha$ there are infinitely many rational numbers $m/n$ with

$$|\alpha - \frac{m}{n}| < \frac{1}{n^2}.$$

Although the proof of this fact is not especially difficult, it will not be given here. The interested reader may attempt a proof of his own, or consult reference [4] of the Suggested Reading, which proves this result and many related ones.

10. Let $F = \int_0^x f$ and $G = \int_0^x f^{-1}$. Then Problem 13-16 states that

$$G(x) = \int_0^x f^{-1} = x\, f^{-1}(x) - \int_0^{f^{-1}(x)} f$$

$$= x\, f^{-1}(x) - F(f^{-1}(x)).$$

Consequently, $G'(x) = f^{-1}(x)$.

11. (a) $F'(x) = 1/x$; $G'(x) = (1/bx)\cdot b = 1/x$.

(b) It follows from part (a) that there is some $c$ such that $F(x) = G(x) + c$ for all $x > 0$. Since $F(1) = 0 = G(1)$, we have $F(x) = G(x)$ for all $x > 0$.

12. Suppose f is continuous on [a, b] and $f(a) < 0 < f(b)$. The Fundamental Theorem of Calculus shows that $f = F'$ for some F (namely $F(x) = \int_0^x f$). Darboux's Theorem then implies that $f(x) = 0$ for some x in [a, b].

13. We have

$$F(x) = \int_{f(x)}^{a} h(t)\,dt + \int_{a}^{g(x)} h(t)\,dt,$$

so

$$F'(x) = -h(f(x))\cdot f'(x) + h(g(x))\cdot g'(x).$$

14. (a) Equation (*) just says that $(G\circ y)' = F'$ in the interval, so there is a c such that $G\circ y = F + c$, in this interval, i.e., $G(y(x)) = F(x) + c$ for all x in the interval.

(b) Conversely, if y satisfies (**), then differentiation yields (*).

(c) If

$$y'(x) = \frac{1 + x^2}{1 + y(x)},$$

so that

$$[1 + y(x)]y'(x) = 1 + x^2,$$

then there is some c such that

$$y(x) + \frac{y^2(x)}{2} = x + \frac{x^3}{3} + c$$

for all x in the interval on which y is defined. So

$$y^2(x) + 2y(x) - 2x - \tfrac{2}{3}x^3 - c = 0, \qquad \text{(calling 2c simply c)}$$

so

$$y(x) = \frac{-2 + \sqrt{4 + 4(x + \frac{2}{3}x^3 + c)}}{2}$$

or

$$y(x) = \frac{-2 - \sqrt{4 + 4(x + \frac{2}{3}x^3 + c)}}{2}.$$

These solutions are never defined on all of **R** , since $4 + 4(x + 2x^3/3 + c) < 0$ for $x < 0$ with $|x|$ sufficiently large.

(d) If

$$(1 + 5[y(x)]^4)y'(x) = -1,$$

then there is some constant c such that

$$[y(x)]^5 + y(x) + x = c.$$

(e) If $y(x)y'(x) = -x$, then there is some c such that

$$\frac{[y(x)]^2}{2} = \frac{-x^2 + c}{2},$$

so

$$y(x) = \sqrt{c - x^2}$$

or

$$y(x) = -\sqrt{c - x^2}.$$

If $y(0) = -1$, then clearly $y(x) = -\sqrt{1 - x^2}$ (for $|x| < 1$).

15. (a)

$$\int_1^\infty x^r\,dx = \lim_{N\to\infty} \frac{N^{r+1}}{r+1} - \frac{1}{r+1} = \frac{-1}{r+1}$$

(because $r + 1 < 0$, so $\lim_{N\to\infty} N^{r+1} = 0$).

(b) Problem 13-12 implies that

$$\int_1^{2^n} \frac{1}{x}\,dx = \underbrace{\int_1^2 \frac{1}{x}\,dx + \dots + \int_1^2 \frac{1}{x}\,dx}_{n \text{ times}}$$
$$= n\int_1^2 \frac{1}{x}\,dx.$$

Since $\int_1^2 1/x\,dx > 0$, we have $\lim_{n\to\infty} \int_1^{2^n} 1/x\,dx = \infty$.

(c) The function $I(N) = \int_0^N g$ is clearly increasing, and it is bounded above by $\int_0^\infty f$. Consequently, $\lim_{N\to\infty} I(N)$ exists (it is the least upper bound of $\{I(N) : N \geq 0\}$).

(d) Clearly $\int_0^\infty 1/(1 + x^2)\,dx$ exists if $\int_1^\infty 1/(1 + x^2)\,dx$ exists; the latter integral exists by part (c), because $\int_1^\infty 1/x^2\,dx$ exists, by part (a), and $1/(1 + x^2) \leq 1/x^2$.

16. (a) Clearly $\int_{-\infty}^{0} 1/(1 + x^2)\,dx$ exists; in fact, it equals $\int_{0}^{\infty} 1/(1 + x^2)\,dx$.

(b) $\int_{0}^{\infty} x\,dx$ does not exist.

(c) If $\lim_{N\to\infty} h(N) = \infty$ and $\lim_{N\to-\infty} g(N) = -\infty$ and $\int_{-\infty}^{\infty} f$ exists, then

$$\lim_{N\to\infty} \int_{g(N)}^{h(N)} f = \int_{-\infty}^{\infty} f.$$

Proof. Given $\epsilon > 0$, choose $M_0$ so that $\left|\int_{0}^{\infty} f - \int_{0}^{M} f\right| < \epsilon/2$ and $\left|\int_{-\infty}^{0} f - \int_{-M}^{0} f\right| < \epsilon/2$ for all $M > M_0$. Now choose $N$ so that $h(N) > M$ and $g(N) < -M$ for all $N > N_0$. Then for $N > N_0$ we have

$$\left|\int_{-\infty}^{\infty} f - \int_{g(N)}^{h(n)} f\right| \le \left|\int_{0}^{\infty} f - \int_{0}^{h(N)} f\right| + \left|\int_{-\infty}^{0} f - \int_{g(N)}^{0} f\right| < \epsilon/2 + \epsilon/2 = \epsilon.$$

17. (a)

$$\lim_{\epsilon\to0^+} \int_{\epsilon}^{a} \frac{1}{\sqrt{x}}\,dx = \lim_{\epsilon\to0^+} 2\sqrt{a} - 2\sqrt{\epsilon} = 2\sqrt{a}.$$

(b)

$$\int_{0}^{a} x^r\,dx = \lim_{\epsilon\to0^+} \int_{\epsilon}^{a} x^r\,dx = \lim_{\epsilon\to0^+} \frac{a^{r+1}}{r + 1} - \frac{\epsilon^{r+1}}{r + 1} = \frac{a^{r+1}}{r + 1}$$

(because $r + 1 > 0$, so $\lim_{\epsilon\to0^+} \epsilon^{r+1} = 0$).

(c) Problem 13-12 implies that

$$\underbrace{\int_{1/2^n}^{1} \frac{1}{x}\,dx + \ldots + \int_{1/2^n}^{1} \frac{1}{x}\,dx}_{\text{n times}} = \int_{2}^{1} \frac{1}{x}\,dx,$$

so

$$\int_{1/2^n}^{1} \frac{1}{x}\,dx = -n \int_{1}^{2} \frac{1}{x}\,dx,$$

so $\lim_{\epsilon \to 0^+} \int_{\epsilon}^{1} 1/x\,dx$ does not exist. Of course, this implies that $\lim_{\epsilon \to 0^+} \int_{\epsilon}^{a} 1/x\,dx$ does not exist for any $a > 0$.

(d)

$$\int_{a}^{0} |x|^r\,dx = \lim_{\epsilon \to 0^-} \int_{a}^{\epsilon} |x|^r\,dx$$

$$= -\lim_{\epsilon \to 0^+} \int_{\epsilon}^{a} x^r\,dx$$

$$= -\frac{a^{r+1}}{r+1}.$$

(e) Since $\lim_{x \to 1} 1/\sqrt{1 - x^2} = \lim_{x \to -1} 1/\sqrt{1 - x^2} = \infty$, we define

$$\int_{-1}^{1} \frac{1}{\sqrt{1 - x^2}}\,dx = \int_{-1}^{0} \frac{1}{\sqrt{1 - x^2}}\,dx + \int_{0}^{1} \frac{1}{\sqrt{1 - x^2}}\,dx$$

$$= \lim_{\epsilon \to -1^+} \int_{\epsilon}^{0} \frac{1}{\sqrt{1 - x^2}}\,dx + \lim_{\epsilon \to 1^-} \int_{0}^{\epsilon} \frac{1}{\sqrt{1 - x^2}}\,dx$$

$$= 2 \lim_{\epsilon \to -1^+} \int_{\epsilon}^{0} \frac{1}{\sqrt{1 - x^2}}\,dx.$$

Now the limit

$$\lim_{\epsilon \to -1^+} \int_\epsilon^0 \frac{1}{\sqrt{1 + x}}\, dx = \lim_{\epsilon \to 0^+} \int_\epsilon^1 \frac{1}{\sqrt{x}}\, dx$$

exists by part (a). For $-1 < x < 0$ we have

$$x(1 + x) < 0,$$

$$x < -x^2,$$

$$1 + x < 1 - x^2,$$

$$\sqrt{1 + x} < \sqrt{1 - x^2},$$

$$\frac{1}{\sqrt{1 + x}} > \frac{1}{\sqrt{1 - x^2}}.$$

It follows that

$$\lim_{\epsilon \to -1^+} \int_\epsilon^0 \frac{1}{\sqrt{1 - x^2}}\, dx$$

also exists.

18. (a)

$$\int_0^\infty f(x)\, dx = \lim_{\epsilon \to 0^+} \int_\epsilon^1 \frac{1}{\sqrt{x}}\, dx + \lim_{N \to \infty} \int_1^N \frac{1}{x^2}\, dx$$

$$= 2 + 1 = 3,$$

by Problem 17(a) and 15(a).

(b) If $-1 < r < 0$, then $\int_1^\infty x^r\,dx$ does not exist, since $x^r \geq x^{-1}$ for $x \geq 1$ and $\int_1^\infty x^{-1}\,dx$ does not exist. If $x < -1$, then $\int_0^1 x^r\,dx$ does not exist, since $x^r \geq x^{-1}$ for $0 < x \leq 1$ and $\int_0^1 x^{-1}\,dx$ does not exist. (Of course, if $r > 0$, then $\int_1^\infty x^r\,dx$ does not exist.)

# CHAPTER 15

1. (ii) $$\frac{1}{\sqrt{1-[\arctan(\arccos x)]^2}}\cdot\frac{1}{1+(\arccos x)^2}\cdot\frac{-1}{\sqrt{1-x^2}}.$$

(iv) $$\frac{1}{\sqrt{1-\left(\frac{1}{\sqrt{1+x^2}}\right)^2}}\cdot\frac{-x}{(1+x^2)^{3/2}}$$

$$=\frac{1}{\sqrt{\frac{x^2}{1+x^2}}}\cdot\frac{-x}{(1+x^2)^{3/2}}$$

$$=\frac{-1}{1+x^2}.$$

(This result is not surprising, since $f(x) = \arctan 1/x = \pi/2 - \arctan x$).

2. (ii)

$$\lim_{x\to 0}\frac{\sin x - x + x^3/6}{x^4} = \lim_{x\to 0}\frac{\cos x - 1 + x^2/2}{4x^3}$$

$$= \lim_{x\to 0}\frac{-\sin x + x}{12x^2} = \lim_{x\to 0}\frac{-\cos x + 1}{24x} = \lim_{x\to 0}\frac{\sin x}{24} = 0.$$

(iv)

$$\lim_{x\to 0}\frac{\cos x - 1 + x^2/2}{x^4} = \lim_{x\to 0}\frac{-\sin x + x}{4x^3}$$

$$= \lim_{x\to 0}\frac{-\cos x + 1}{12x^2} = \lim_{x\to 0}\frac{\sin x}{24x} = \lim_{x\to 0}\frac{\cos x}{24} = \frac{1}{24}.$$

(vi)

$$\lim_{x \to 0}\left(\frac{1}{x} - \frac{1}{\sin x}\right) = \lim_{x \to 0} \frac{\sin x - x}{x \sin x}$$

$$= \lim_{x \to 0} \frac{\cos x - 1}{\sin x + x \cos x} = \lim_{x \to 0} \frac{-\sin x}{2 \cos x - x \sin x} = 0.$$

3. (a)

$$f'(0) = \lim_{h \to 0} \frac{\frac{\sin h}{h} - 1}{h} = \lim_{h \to 0} \frac{\sin h - h}{h^2}$$

$$= \lim_{h \to 0} \frac{\cos h - 1}{h} = 0.$$

(b) Since

$$f'(x) = \frac{x \cos x - \sin x}{x^2} \qquad \text{for } x \neq 0,$$

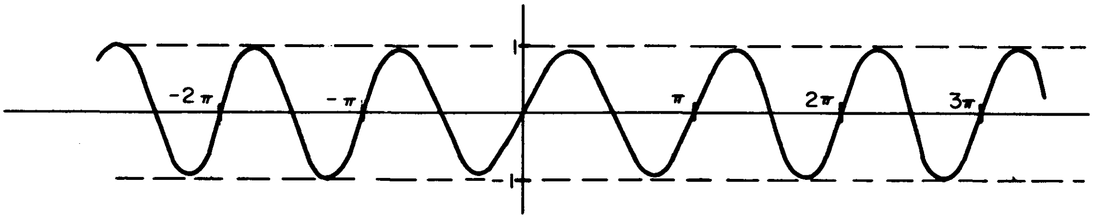
we have

$$f''(0) = \lim_{h \to 0} \frac{h \cos h - \sin h}{h^3}$$

$$= \lim_{h \to 0} \frac{\cos h - h \sin h - \cos h}{3h^2}$$

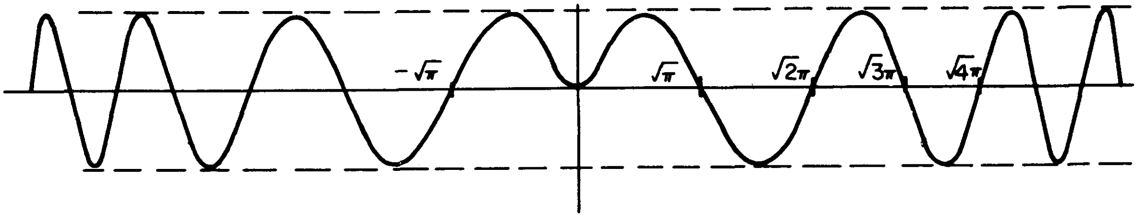
$$= -\frac{1}{3}.$$

4. (a)



(b) Clearly $f(x) = 0$ for $x = \sqrt{k\pi}$. The numbers $\sqrt{k\pi}$ become arbitrarily large, of course (since $\sqrt{k^2\pi} > k$), but they also cluster closer and closer together (because, for example,

$$\begin{aligned}\sqrt{k+1} - \sqrt{k} &= \frac{1}{2\sqrt{x}} \qquad \text{for some } x \text{ in } (k, k+1), \text{ by the Mean Value Theorem} \\ &< \frac{1}{2\sqrt{k}} ).\end{aligned}$$



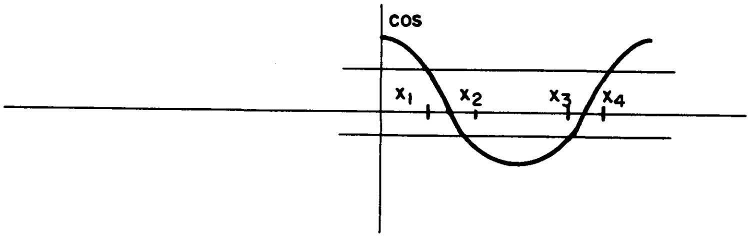
(c) If

$$\begin{aligned}0 = f'(x) &= \cos x + 2\cos 2x = \cos x + 2(\cos^2 x - [1 - \cos^2 x]) \\ &= \cos x + 2(2\cos^2 x - 1) \\ &= 4\cos^2 x + \cos x - 2,\end{aligned}$$

then

$$\cos x = \frac{-1 \pm \sqrt{1 + 32}}{8} = \frac{-1 \pm \sqrt{33}}{8}.$$

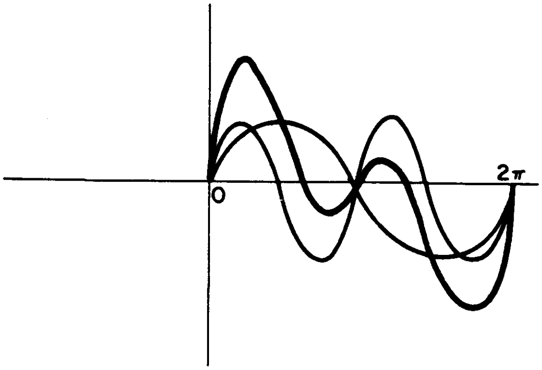
Since $0 < [-1 + \sqrt{33}]/8 < 1$ and $-1 < [-1 - \sqrt{33}]/8 < 0$, there will be four such $x$ in $[0, 2\pi]$:



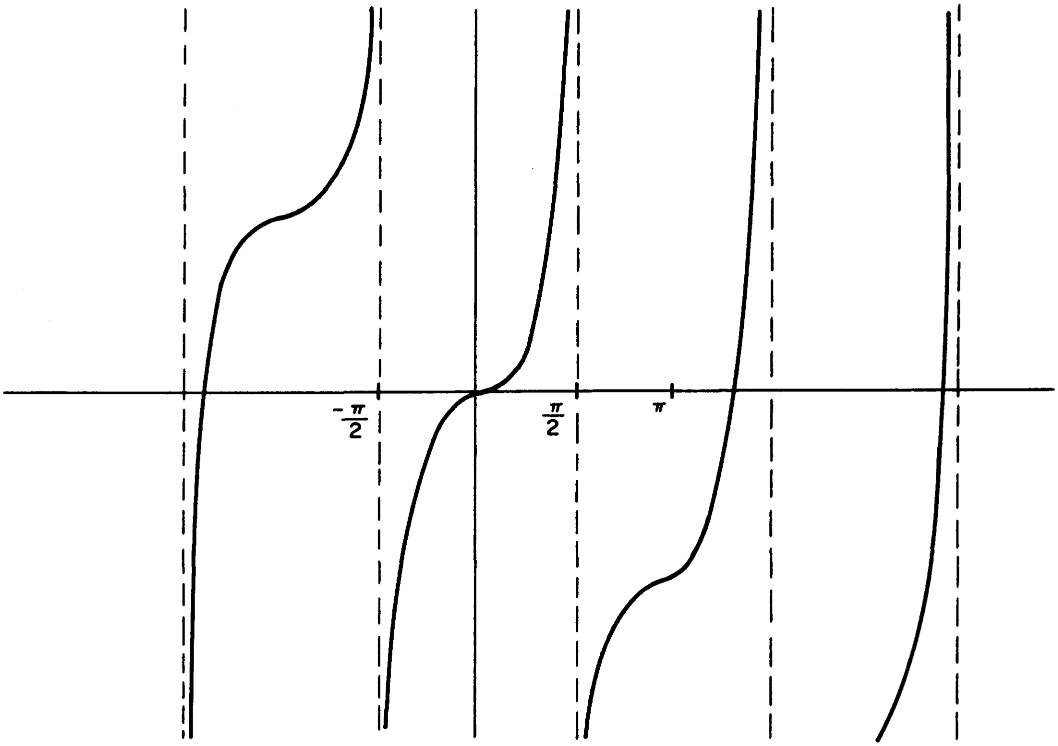
The critical points $x_1$ and $x_4$, with $\cos x_1 = \cos x_4 = [-1 + \sqrt{33}]/8$, satisfy $0 < x_1 < \pi/2$ and $3\pi/4 < x_4 < 2\pi$; so $f(x_1) > 0$ and $f(x_2) < 0$, since sin x and sin 2x are both positive on $(0, \pi/2)$ and both negative on $(3\pi/4, 1)$. To determine the sign of $f(x_3)$ and $f(x_4)$ notice that

$$\begin{aligned} f(x) &= \sin x + \sin 2x \\ &= \sin x + 2 \sin x \cos x \\ &= \sin x\,(1 + 2 \cos x). \end{aligned}$$

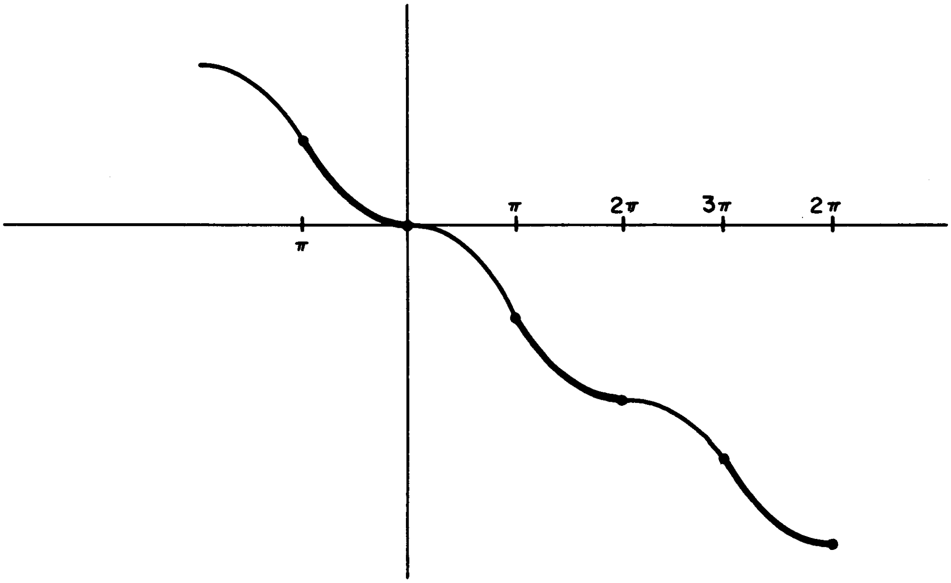
Now $\sin(x_2) > 0$, since $0 < x_2 < \pi$, but $1 + 2\cos(x_2) < 0$, so $f(x_2) < 0$. Similarly, $f(x_3) > 0$.



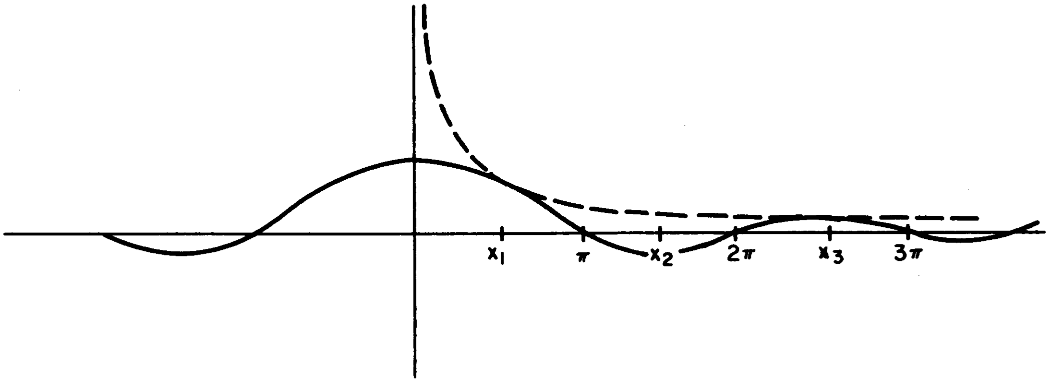
(d) $f'(x) = \sec^2 x - 1 = \tan^2 x \geq 0$ for all $x$, so $f$ is always increasing. On $(-\pi/2, \pi/2)$ clearly $f$ increases from $-\infty$ to $\infty$. On $(k\pi - \pi/2, k\pi + \pi/2)$ the derivative $f'$ is the same as on $(-\pi/2, \pi/2)$, so $f$ differs by a constant from $f$ on $(-\pi/2, \pi/2)$. The constant is clearly $-\pi$.



(e) $f'(x) = \cos x - 1 \leq 0$ for all $x$, so $f$ is decreasing. Moreover $f'$ is periodic, so $f$ is the same on $[2\pi, 4\pi]$ as on $[0, 2\pi]$ except moved down by $2\pi$. Since $f''(x) = -\sin x$, it follows from the Appendix to Chapter 11 that $f$ is concave on $[0, \pi]$ and convex on $[\pi, 2\pi]$.



(f) If $0 = f'(x) = (x \cos x - \sin x)/x^2$, then $x = \tan x$. The graph in part (d) shows that on the right side of the vertical axis this happens for $0 < x_1 < x_2 < \ldots$, where $x_n$ is slightly smaller than $n\pi + \pi/2$.



5. For any particular number y, define $f(x) = \cos(x + y)$. Then

$$f'(x) = -\sin(x + y),$$
$$f''(x) = -\cos(x + y),$$

so

$$f'' + f = 0,$$
$$f(0) = \cos y$$
$$f'(0) = -\sin y.$$

So

$$f = (-\sin y)\cdot\sin + (\cos y)\cdot\cos,$$

so

$$\cos(x + y) = \cos y \cos x - \sin y \sin x.$$

7. (a) Clearly $f(x) = A \sin(x + B)$ satisfies $f + f'' = 0$. (Moreover, $a = f'(0) = A \cos B$ and $b = f(0) = A \sin B$.)

(b) It clearly suffices to choose A and B so that $a = A \cos B$ and $b = A \sin B$. Since we want

$$a^2 + b^2 = (A \cos B)^2 + (A \sin B)^2$$

we must clearly choose

$$A = \sqrt{a^2 + b^2}.$$

If $a \neq 0$, we can choose

$$B = \arctan \frac{b}{a}.$$

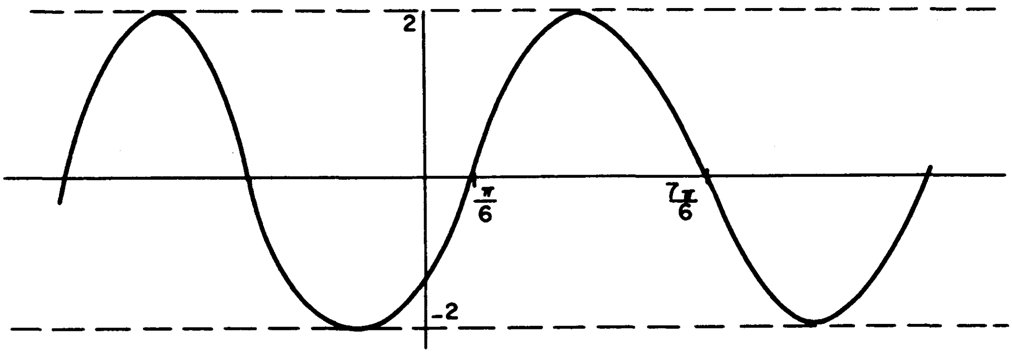
If $a = 0$, we can choose $B = \pi/2$.

(c) $\sqrt{3} \sin x + \cos x = A \sin(x + B)$, where

$$A = \sqrt{(\sqrt{3})^2 + 1} = 2$$

$$B = \arctan \frac{1}{\sqrt{3}} = \pi/6,$$

so $\sqrt{3} \sin x + \cos x = 2 \sin(x + \pi/6)$.



9. From the addition formula for sin we obtain, for $|\alpha| \leq 1$ and $|\beta| \leq 1$,

$$\begin{aligned}\sin(\arcsin \alpha + \arcsin \beta) &= \sin(\arcsin \alpha) \cos(\arcsin \beta) \\ &\quad + \cos(\arcsin \alpha) \sin(\arcsin \beta) \\ &= \alpha \sqrt{1 - \beta^2} + \beta \sqrt{1 - \alpha^2}.\end{aligned}$$

Consequently

$$\arcsin \alpha + \arcsin \beta = \arcsin(\alpha\sqrt{1-\beta^2} + \beta\sqrt{1-\alpha^2}),$$

provided that $-\pi/2 \le \arcsin \alpha + \arcsin \beta \le \pi/2$. (If $\pi/2 < \arcsin \alpha + \arcsin \beta \le \pi$, we must replace the right side by $\pi - \arcsin(\alpha\sqrt{1-\beta^2} + \beta\sqrt{1-\alpha^2})$, and if $-\pi \le \arcsin \alpha + \arcsin \beta < \frac{\pi}{2}$, we must replace it by $-\pi - \arcsin(\alpha\sqrt{1-\beta^2} + \beta\sqrt{1-\alpha^2})$.)

12. (a) If

$$H(a) = \int_{-\pi}^{\pi} (f(x) - a\cos nx)^2\,dx$$

$$= a^2\int_{-\pi}^{\pi} \cos^2 nx\,dx - 2a\int_{-\pi}^{\pi} f(x)\cos nx\,dx + \int_{-\pi}^{\pi} f(x)^2\,dx,$$

then the minimum occurs for

$$0 = H'(a) = 2a\int_{-\pi}^{\pi} \cos^2 nx\,dx - 2\int_{-\pi}^{\pi} f(x)\cos nx\,dx,$$

so

$$a = \frac{\int_{-\pi}^{\pi} f(x)\cos nx\,dx}{\int_{-\pi}^{\pi} \cos^2 nx\,dx} = \int_{-\pi}^{\pi} f(x)\cos nx\,dx,$$

by Problem 11. The proof for sin nx is similar.

(b)

$$\int_{-\pi}^{\pi} \left(f(x) - \left[\frac{c_0}{2} + \sum_{n=1}^{N} c_n \cos nx + d_n \sin nx\right]\right)^2 dx$$

$$= \int_{-\pi}^{\pi} [f(x)]^2\, dx - 2\int_{-\pi}^{\pi} f(x)\left[\frac{c_0}{2} + \sum_{n=1}^{N} c_n \cos nx + d_n \sin nx\right] dx$$

$$+ \int_{-\pi}^{\pi} \left[\frac{c_0^{\,2}}{4} + \sum_{n=1}^{N} c_n^{\,2} \cos^2 nx + d_n^{\,2} \sin^2 nx\right] dx$$

$$+ \int_{-\pi}^{\pi} \sum_{n,m=1}^{N} c_n d_m \cos nx \sin mx\, dx$$

$$+ \int_{-\pi}^{\pi} \frac{c_0}{2} \sum_{n=1}^{N} c_n \cos nx + d_n \sin nx\, dx$$

$$= \int_{-\pi}^{\pi} [f(x)]^2\, dx - 2\pi\left(\frac{a_0 c_0}{2} + \sum_{n=1}^{N} a_n c_n + b_n d_n\right) + \pi\left(\frac{c_0^{\,2}}{2} + \sum_{n=1}^{N} c_n^{\,2} + d_n^{\,2}\right),$$

using Problem 11, the definition of $a_n$ and $b_n$, and the fact that the last integral vanishes because $\int_{-\pi}^{\pi} \cos nx\, dx = \int_{-\pi}^{\pi} \sin nx\, dx = 0$. The second equality follows by algebra.

13. Substituting $a = (x + y)/2$, $b = (x - y)/2$ in

$$\sin(a + b) + \sin(a - b) = \sin a \cos b + \cos a \sin b + \sin a \cos(-b) + \cos a \sin(-b$$

$$= 2 \sin a \cos b$$

yields

$$\sin x + \sin y = 2 \sin\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right).$$

Similarly, from

$$\begin{aligned}\cos(a + b) + \cos(a - b) &= \cos a \cos b - \sin a \sin b + \cos a \cos(-b) - \sin a \sin(-b)\\ &= 2 \cos a \cos b\end{aligned}$$

we obtain

$$\cos x + \cos y = 2 \cos\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right).$$

15. If $y = \arctan x$, then

$$x = \tan y = \frac{\sin y}{\cos y} = \frac{\sin y}{\sqrt{1 - \sin^2 y}},$$

so

$$\begin{aligned}x\sqrt{1 - \sin^2 y} &= \sin y,\\ x^2(1 - \sin^2 y) &= \sin^2 y\\ \sin^2 y &= \frac{x^2}{\sqrt{1 + x^2}},\end{aligned}$$

so

$$\sin(\arctan x) = \sin y = \frac{x}{\sqrt{1 + x^2}},$$

$$\cos(\arctan x) = \cos y = \sqrt{1 - \sin^2 y} = \frac{1}{\sqrt{1 + x^2}}.$$

16. If $x = \tan u/2$, then $u = 2 \arctan x$, so by Problem 15

$$\begin{aligned} \sin u &= \sin(2 \arctan x) \\ &= 2 \sin(\arctan x)\cos(\arctan x) \\ &= \frac{2x}{1 + x^2}, \\ \cos u &= \sqrt{1 - \sin^2 u} = \frac{1 - x^2}{1 + x^2}. \end{aligned}$$

17. (a) By the addition formula,

$$\sin(x + \pi/2) = \sin x \cos \pi/2 + \cos x \sin \pi/2 = \cos x.$$

(b) Part (a) implies that $x + \pi/2 = \arcsin(\cos x)$ for $-\pi/2 \leq x + \pi/2 \leq \pi/2$, or equivalently $-\pi \leq x \leq 0$. If $x = 2k\pi + x'$ for $-\pi \leq x \leq 0$, then $\cos x = \cos x'$, and if $x = 2k\pi + x'$ for $0 \leq x' \leq \pi$, then $\cos x = \cos x' = \cos(-x')$. So

$$\arcsin(\cos x) = \begin{cases} x - 2k\pi + \pi/2, & (2k - 1)\pi \leq x \leq 2k\pi \\ 2k\pi + \pi/2 - x, & 2k\pi \leq x \leq (2k + 1)\pi. \end{cases}$$

Similarly, from

$$\cos(x - \pi/2) = \sin x,$$

we conclude that

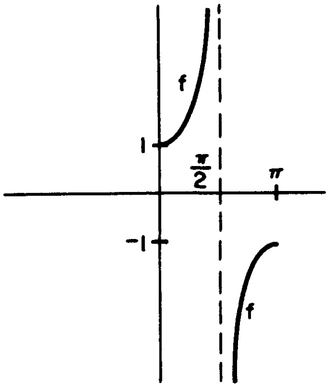
$$\arccos(\sin x) = \begin{cases} x - 2k\pi - \pi/2, & 2k\pi + \pi/2 \le x \le (2k + 1)\pi + \pi/2 \\ (2k + 1)\pi - \pi/2 - x, & 2k\pi - \pi/2 \le x \le 2k\pi + \pi/2. \end{cases}$$

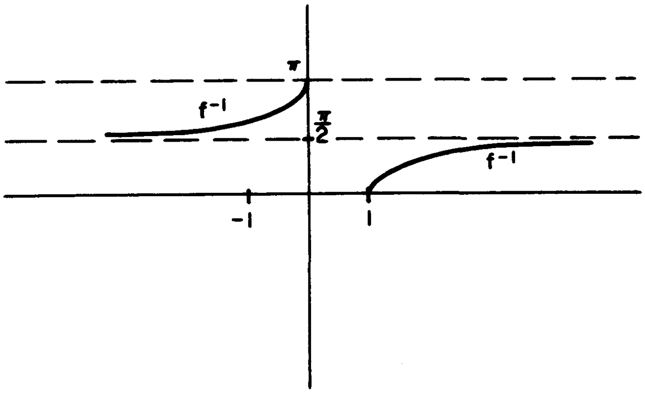
21. If $(x, y)$ is on the unit circle, then $x^2 + y^2 = 1$. In particular $|x^2| \le 1$, so $-1 \le x \le 1$. On the intervals $[0, \pi]$ and $[-\pi, 0]$ the function cos takes on all values between -1 and 1, so there is some $\theta$ in $[0, \pi]$ with $x = \cos\theta$, and also some $\theta$ in $[-\pi, 0]$ with $x = \cos\theta$. If $y \ge 0$, then $y = \sin\theta$ when is in $[0, \pi]$, and if $y \le 0$, then $y = \sin\theta$ when $\theta$ is in $[-\pi, 0]$.

22. (a) If $a < 2k\pi + \pi/2 < b$, then sin is not one-one on $[a, b]$, because sin has a maximum at $2k\pi + \pi/2$, so sin takes on all values slightly less than 1 on both sides of $2k\pi + \pi/2$. Similarly, we cannot have $a < 2k\pi - \pi/2 < b$. Since the numbers of the form $2k\pi \pm \pi/2$ are within $\pi$ of each other, $\pi$ is the maximum length of an interval $[a, b]$ on which sin is one-one, and in this case $[a, b]$ must be of the form $[2k\pi - \pi/2, 2k\pi + \pi/2]$ or $[2k\pi + \pi/2, 2(k + 1)\pi - \pi/2]$.

(b) $(g^{-1})'(x) = 1/\sqrt{1 - x^2}$, since $g^{-1}(x) = \arcsin x + 2k\pi$.

23. The domain of $f^{-1}$ is $(-\infty, 1] \cup [1, \infty)$.





24. By the Mean Value Theorem,

$$|\sin x - \sin y| = |x - y|\cdot|\cos\ \theta| \qquad \text{for some } \theta \text{ between } x \text{ and } y$$

$$\leq |x - y|.$$

Strict inequality holds unless $\theta = 2k\pi$. But in any case, if $x < y$, say, then we can choose $x < z < y$ so that $(x, z)$ does not contain any number of the form $2k\pi$. Then

$$\sin y - \sin x = (\sin y - \sin z) + (\sin z - \sin x)$$

$$= (y - z)\cos\theta_1 + (z - x)\cos\theta_2$$

for some $\theta_1$ in $(y, z)$ and $\theta_2$ in $(x, z)$. Since $|\cos\theta_1| \leq 1$ and $|\cos\theta_2| < 1$, it follows that

$$|\sin y - \sin x| < |y - x|.$$

25. (a)

$$\lim_{\lambda\to\infty}\int_c^d \sin\lambda x\,dx = \lim_{\lambda\to\infty}\frac{\cos\lambda c}{\lambda} - \frac{\cos\lambda d}{\lambda} = 0.$$

(b) If s has the values $s_i$ on $(t_{i-1}, t_i)$, then

$$\lim_{\lambda\to\infty}\int_a^b s(x)\sin\lambda x\,dx = \lim_{\lambda\to\infty}\sum_{i=1}^{n} s_i \int_{t_{i-1}}^{t_i} \sin\lambda x\,dx$$

$$= 0, \text{ by part (a).}$$

(c) For any $\epsilon > 0$ there is, by Problem 13-17, a step function $s \le f$ with

$$\int_a^b [f(x) - s(x)]\,dx < \epsilon.$$

Now

$$\left|\int_a^b f(x)\sin\lambda x\,dx - \int_a^b s(x)\sin\lambda x\,dx\right| = \left|\int_a^b [f(x) - s(x)]\sin\lambda x\,dx\right|$$

$$\le \int_a^b [f(x) - s(x)]\,|\sin\lambda x|\,dx$$

$$\le \int_a^b [f(x) - s(x)]\,dx < \epsilon.$$

Part (b) then shows that

$$\lim_{\lambda\to\infty}\left|\int_a^b f(x)\sin\lambda x\,dx\right| < \epsilon.$$

Since this is true for every $\epsilon > 0$, the limit must be 0.

26. (a) We have

$$\text{area } OAB < \frac{x}{2} < \text{area } OCB,$$

so

$$\frac{\sin x}{2} < \frac{x}{2} < \frac{\sin x}{2 \cos x}.$$

(b) From

$$\frac{\sin x}{2} < \frac{x}{2}$$

we obtain

$$\frac{\sin x}{x} < 1;$$

from

$$\frac{x}{2} < \frac{\sin x}{2 \cos x}$$

we obtain

$$\cos x < \frac{\sin x}{x}.$$

Since $\lim_{x \to 0} \cos x = 1$, it follows that $\lim_{x \to 0} (\sin x)/x = 1$.

(c)

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = \lim_{x \to 0} \frac{1 - \cos^2 x}{x(1 + \cos x)}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{1 + \cos x} = 1 \cdot 0 = 0.$$

(d)

$$\begin{aligned}\sin'(x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\ &= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \lim_{h\to 0} \frac{\sin h}{h} \cos x + \lim_{h\to 0} \frac{\cos h - 1}{h} \sin x \\ &= \cos x.\end{aligned}$$

27. (a) Clearly $\alpha$ is odd and increasing. The limit $\lim_{x\to\infty} \alpha(x)$, i.e., the improper integral $\int_0^\infty (1+t^2)^{-1}\,dt$, exists by Problem 14-15.

(b)

$$\begin{aligned}(\alpha^{-1})'(x) &= \frac{1}{\alpha'(\alpha^{-1}(x))} \\ &= \frac{1}{\dfrac{1}{1+[\alpha^{-1}(x)]^2}} = 1 + [\alpha^{-1}(x)]^2.\end{aligned}$$

(c) If $-\pi/2 < x < \pi/2$, then

$$\cos x = \frac{1}{\sqrt{1+[\alpha^{-1}(x)]^2}} = (1 + [\alpha^{-1}(x)]^2)^{-1/2},$$

so

$$\begin{aligned}\cos'(x) &= -\alpha^{-1}(x)(\alpha^{-1})'(x)(1 + [\alpha^{-1}(x)]^2)^{-3/2}\\ &= -\alpha^{-1}(x)(1 + [\alpha^{-1}(x)]^2)^{-1/2}\\ &= -\tan x \cos x.\end{aligned}$$

Naturally the same result holds if x is not of the form $k\pi + \pi/2$ or $k\pi - \pi/2$. (For x which are of this form we have, by Theorem 11-7,

$$\begin{aligned}\cos'(x) &= \lim_{y\to x} \cos'(y)\\ &= \lim_{y\to x} -\tan y \cos y\\ &= \lim_{y\to x} \frac{-\tan y}{\sqrt{1 + \tan^2 y}}\\ &= -1, \qquad \text{since } \lim_{y\to x} \tan x = \infty.)\end{aligned}$$

Now for x not of the form $k\pi + \pi/2$ or $k\pi - \pi/2$ we have

$$\begin{aligned}\cos''(x) &= -\tan x \cos'(x) - \tan'(x) \cos x\\ &= -\tan^2 x \cos x - [1 + \tan^2 x] \cos x \qquad \text{by part (b)}\\ &= -\cos x.\end{aligned}$$

For x which are of this form we have

$$\cos''(x) = \lim_{y \to x} \cos''(y) = \lim_{y \to x} -\cos y,$$

$$= \lim_{y \to x} \frac{-1}{\sqrt{1 + \tan^2 y}} = 0, \qquad \text{since } \lim_{y \to x} \tan y = \infty.$$

28. (a) $(y_0{}^2 + (y_0')^2)' = 2y_0y_0' + 2y_0'y_0'' = 2y_0'(y_0' + y_0'') = 0$, so $y_0{}^2 + (y_0')^2$ is a constant. The constant is non-zero, since $y_0$ is not always 0, so $y_0(0)^2 + y_0'(0)^2 \neq 0$, so either $y_0(0) \neq 0$ or $y_0'(0) \neq 0$.

(b) Any function $s = ay_0 + by_0'$ satisfies $s'' + s = 0$, so we just have to choose a and b such that

$$ay_0(0) + by_0'(0) = 0$$

$$ay_0'(0) - by_0(0) = 1.$$

This is always possible, since

$$-y_0(0)^2 - y_0'(0)^2 \neq 0.$$

(c) Suppose that $\cos x > 0$ for all $x > 0$. Then sin would be increasing, since $\sin' = \cos$. Since $\sin 0 = 0$, this would mean that $\sin x > 0$ for all $x > 0$. Thus we would have $\cos' = -\sin x < 0$ for all $x > 0$, so cos would be decreasing. Thus cos would satisfy all the hypotheses for f in Problem 6 of the Appendix to Chapter 11. But then the problem implies that $\cos''(x) = -\cos x = 0$ for some $x > 0$, a contradiction.

(d) Since $\cos x > 0$ for $0 < x < x_0 = \pi/2$, the function sin is increasing on $[0, \pi/2]$. Since $\sin 0 = 0$, it follows that $\sin \pi/2 > 0$, so $\sin \pi/2 = 1$.

(e)

$$\cos \pi = \cos(\pi/2 + \pi/2) = \cos^2 \pi/2 - \sin^2 \pi/2 = 0 - 1 = -1.$$

$$\sin \pi = \sin(\pi/2 + \pi/2) = 2 \sin \pi/2 \cos \pi/2 = 0.$$

$$\cos 2\pi = \cos(\pi + \pi) = \cos^2 \pi - \sin^2 \pi = 1.$$

$$\sin 2\pi = \sin(\pi + \pi) = 2 \sin \pi \cos \pi = 0.$$

(f)

$$\sin(x + 2\pi) = \sin x \cos 2\pi + \cos x \sin 2\pi = \sin x.$$

$$\cos(x + 2\pi) = \cos x \cos 2\pi - \sin x \sin 2\pi = \cos x.$$

29. (a) A rational function cannot be 0 at infinitely many points unless it is 0 everywhere.

(b) The assumed equation implies that $f_0(x) = 0$ for $x = 2k\pi$, so $f_0 = 0$. So

$$(\sin x)[(\sin x)^{n-1} + f_{n-1}(x)(\sin x)^{n-2} + \dots + f_1(x)] = 0.$$

The term in brackets is continuous and 0 except perhaps at multiples of $2\pi$, so it is 0 everywhere. We have just shown that if sin does not satisfy such an equation for $n - 1$, then it does not satisfy it for $n$.

Since it clearly does not satisfy such an equation for $n = 1$, it does not satisfy it for any $n$.

30. (a) Multiplying the equation for $g_1$ by $\phi_2$ and the equation for $g_2$ by $\phi_1$ we obtain

$$\phi_1''\phi_2 + g_1\phi_1\phi_2 = 0,$$

$$\phi_2''\phi_1 + g_2\phi_1\phi_2 = 0.$$

Subtraction yields the desired equation.

(b)

$$\int_a^b [\phi_1''\phi_2 - \phi_2''\phi_1] = \int_a^b (g_2 - g_1)\phi_1\phi_2 > 0,$$

since $g_2 > g_1$ and $\phi_1\phi_2 > 0$ by assumption.

Since

$$(\phi_1'\phi_2 - \phi_1\phi_2')' = \phi_1''\phi_2 + \phi_1'\phi_2' - \phi_1'\phi_2' - \phi_1\phi_2''$$
$$= \phi_1''\phi_2 - \phi_1\phi_2'',$$

we have

$$0 < \int_a^b [\phi_1''\phi_2 - \phi_2''\phi_1] = [\phi_1'(b)\phi_2(b) - \phi_1(b)\phi_2'(b)] - [\phi_1'(a)\phi_2(a) - \phi_1(a)\phi_2'(a)]$$

$$= \phi_1'(b)\phi_2(b) - \phi_1'(a)\phi_2(a) + [\phi_1(b)\phi_2'(b) - \phi_1(a)\phi_2'(a)].$$

(c) If $\phi_1(a) = \phi_1(b) = 0$, then it follows from part (b) that

$$\phi_1'(b)\phi_2(b) - \phi_1'(a)\phi_2(a) > 0.$$

But clearly

$$\phi_2'(a) \geq 0, \quad \phi_2(b) \geq 0$$

$$\phi_1'(a) \geq 0, \quad \phi_1'(b) \leq 0.$$

This implies that

$$\phi_1'(b)\phi_2(b) - \phi_1'(a)\phi_2(a) \leq 0,$$

a contradiction.

(d) This follows from part (c) by replacing $\phi_1$ by $-\phi_1$ and/or $\phi_2$ by $-\phi_2$.

31. The desired equation is equivalent to

$$\sin(n + \tfrac{1}{2})x + 2 \sin \frac{x}{2} \cos(n + 1)x = \sin(n + \tfrac{3}{2})x$$

$$= \sin(n + 1)x \cos \frac{x}{2} + \cos(n + 1)x \sin \frac{x}{2}$$

and thus to

$$\sin(n + \tfrac{1}{2}x) = \sin(n + 1)x \cos \frac{x}{2} - \sin \frac{x}{2} \cos(n + 1)x,$$

which follows from the addition formula for $\sin([n + 1]x - \frac{1}{2}x)$.

32. (a) If $f(x) = \alpha x + \beta$, then for every P we have

$$\ell(f, P) = \sum_{i=1}^{n} \sqrt{(t_i - t_{i-1})^2 + \alpha^2(t_i - t_{i-1})^2}$$

$$= \sum_{i=1}^{n} (t_i - t_{i-1})\sqrt{1 + \alpha^2}$$

$$= (b - a)\sqrt{1 + \alpha^2},$$

and the distance from $(a, \alpha a + \beta)$ to $(b, \alpha b + \beta)$ is

$$\sqrt{[\alpha(a - b)]^2 + (a - b)^2} = (b - a)\sqrt{1 + \alpha^2}.$$

(b) If $f$ is not linear, then there is some $t$ in $[a, b]$ such that $(a, f(a))$, $(t, f(t))$, and $(b, f(b))$ do not lie on a straight line. Thus if $P = \{a, t, b\}$, then

$$\ell(f, P) = \sqrt{(t - a)^2 + [f(t) - f(a)]^2} + \sqrt{(b - t)^2 + [f(b) - f(t)]^2}$$

$$> \sqrt{(b - a)^2 + [f(b) - f(a)]^2}, \qquad \text{by Problem 4-8.}$$

(c) follows immediately from part (b).

33. (a) For each $i$ there is some $x_i$ in $(t_{i-1}, t_i)$ with

$$f'(x_i)(t_i - t_{i-1}) = f(t_{i-1}) - f(t_i).$$

So

$$L(\sqrt{1 + (f')^2}, P) \leq \sum_{i=1}^{n} (t_i - t_{i-1})\sqrt{1 + [f'(x_i)]^2} \leq U(\sqrt{1 + (f')^2}, P)$$

and

$$\sum_{i=1}^{n} (t_i - t_{i-1})\sqrt{1 + [f'(x_i)]^2} = \sum_{i=1}^{n} \sqrt{(t_i - t_{i-1})^2 + [f'(x_i)(t_i - t_{i-1})]^2}$$

$$= \sum_{i=1}^{n} \sqrt{(t_i - t_{i-1})^2 + [f(t_i) - f(t_{i-1})]^2}$$

$$= \ell(f, P).$$

(b) Since $\sup\{\ell(f, P)\}$ is an upper bound for the set of all $\ell(f, P)$, it is also an upper bound for the set of all $L(\sqrt{1 + (f')^2}, P)$, by part (a).

(c) It suffices to show that

$$\sup\{\ell(f, P)\} \le U(\sqrt{1 + (f')^2}, P'')$$

for any partition $P''$, and to prove this it suffices to show that

$$\ell(f, P') \le U(\sqrt{1 + (f')^2}, P'')$$

for any partition $P'$. If $P$ contains the points of $P'$, then

$$\ell(f, P) \ge \ell(f, P');$$

the proof is similar for the proof for lower sums, putting in one point at a time and using Problem 4-8 to see that this increases $\ell$. Thus if $P$ contains the points of both $P'$ and $P''$, then

$$\ell(f, P') \le \ell(f, P) \le U(\sqrt{1 + (f')^2}, P) \le U(\sqrt{1 + (f')^2}, P'').$$

34. (a) Problem 33 shows that

$$\mathcal{L}(x) = \int_x^1 \sqrt{1 + [f'(t)]^2}\,dt = \int_x^1 \sqrt{1 + \left(\frac{t}{\sqrt{1 - t^2}}\right)^2}\,dt$$

$$= \int_x^1 \frac{1}{\sqrt{1 - t^2}}\,dt.$$

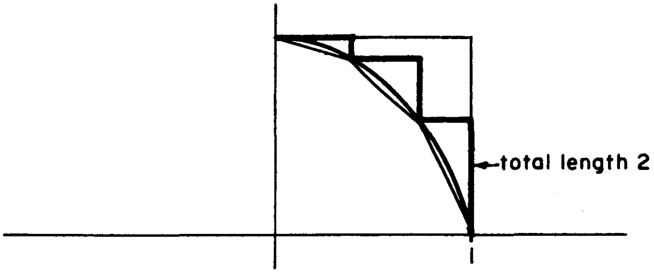
[Actually, a more detailed argument is necessary, because $\int_x^1 1/\sqrt{1 - t^2}dt$ is not an ordinary integral, but an improper integral. It does follow immediately from Problem 33 that

$$\text{length of } f \text{ on } [x, 1 - \epsilon] = \int_x^{1-\epsilon} \frac{1}{\sqrt{1 - t^2}}\,dt.$$

To obtain the desired expression for $\mathcal{L}(x)$ we must then use the fact that

$$\lim_{\epsilon \to 0} (\text{length of } f \text{ on } [x, 1 - \epsilon]) = \text{length of } f \text{ on } [x, 1].$$

This is proved as follows. First of all, the following figure shows that the "length of f on [x, 1]" does make sense; in fact, the length of f on [0, 1] is $\leq 2$.



The same sort of figure also shows that the length of f on $[1 - \epsilon, 1]$ is $\leq 2\epsilon$. The desired limit then follows from this inequality and the fact that

$$\text{length of } f \text{ on } [x, 1 - \epsilon] + \text{length of } f \text{ on } [1 - \epsilon, 1] = \text{length of } f \text{ on } [x, 1].$$

The proof of this latter fact is very similar to the corresponding assertion for integrals.]

(b) This follows from part (a) and the Fundamental Theorem of Calculus.

(c) By the definition given, $\cos = \mathcal{L}^{-1}$, so

$$\cos'(x) = (\mathcal{L}^{-1})'(x) = \frac{1}{\mathcal{L}'(\mathcal{L}^{-1}(x))}$$

$$= \frac{1}{\dfrac{-1}{\sqrt{1 - \cos^2 x}}} = -\sin x.$$

The proof for $\sin'(x)$ is the same as the one in the text.

## CHAPTER 16

1. (a) Let $\Delta$ = area OAB. Since

$$xy = 2\Delta,$$
$$x^2 + y^2 = 1,$$

we have

$$\left(\frac{2\Delta}{y}\right)^2 + y^2 = 1,$$
$$4\Delta^2 + y^4 = y^2,$$
$$y^4 - y^2 + 4\Delta^2 = 0,$$
$$y^2 = \frac{1 \pm \sqrt{1 - 16\Delta^2}}{2}.$$

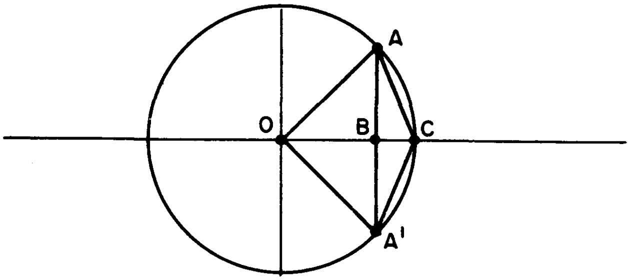
We have

$$y^2 = \frac{1 - \sqrt{1 - 16\Delta^2}}{2},$$

provided that $y^2 < 1/2$, or $y < \sqrt{2}/2$. So

$$\text{area OAC} = \frac{y}{2} = \frac{1}{2}\sqrt{\frac{1 - \sqrt{1 - 16(\text{area OAB})^2}}{2}}.$$

(b) Let $P_m$ be the union of m triangles congruent to the triangle OAA' in the figure below.



Each such triangle has area $A_m/m$ so triangle OAB has area $A_m/2m$. Now $P_{2m}$ is the union of 2m triangles congruent to OAC. So by part (a),

$$A_{2m} = 2m \text{ area } OAC = m\sqrt{\frac{1 - \sqrt{1 - 16(\text{area } OAB)^2}}{2}}$$

$$= m\sqrt{\frac{1 - \sqrt{1 - 16\, A_m^2/4m^2}}{2}}$$

$$= \frac{m}{2}\sqrt{2 - 2\sqrt{1 - (2A_m/m)^2}}.$$

2. (a)

$$\frac{A_m}{A_{2m}} = \frac{2m \text{ area } (OAB)}{2m \text{ area } (OAC)} = OB = \alpha_m.$$

(b)

$$\frac{2}{A_{2^k}} = \frac{2}{A_8}\cdot\frac{A_8}{A_{16}}\cdot\cdots\cdot\frac{A_{2^{k-1}}}{A_{2^k}}$$

$$= \frac{A_4}{A_8}\cdot\cdots\cdot\frac{A_{2^{k-1}}}{A_{2^k}} = \alpha_4\cdot\ldots\cdot\alpha_{2^{k-1}}.$$

(c)

$$\alpha_4 = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} = \sqrt{\frac{1}{2}},$$

$$\alpha_8 = \cos\left(\frac{\pi/4}{2}\right) = \sqrt{\frac{1}{2} + \frac{\cos \pi/4}{2}}$$

$$= \sqrt{\frac{1}{2} + \frac{1}{2}\cdot\sqrt{\frac{1}{2}}},$$

etc.

# CHAPTER 17

1. (ii) $$\frac{1}{1+(\log(1+\log(1+e^{1+e^{1+x}}))}\cdot\frac{1}{1+\log(1+e^{1+e^{1+x}})}$$

$$\cdot\frac{1}{1+e^{1+e^{1+x}}}\cdot e^{1+e^{1+x}}\cdot e^{1+x}.$$

(iv) $\left(\int_0^x e^{-t^2}\,dt\right)\cdot e^{-x^2}$

(vi)

$$f(x) = \frac{\log(\sin x)}{\log e^x} = \frac{\log(\sin x)}{x},$$

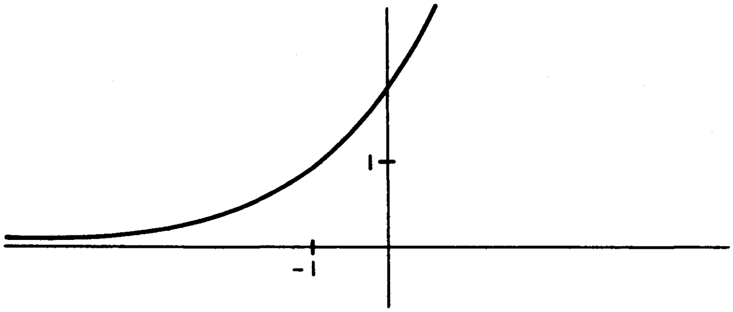
so

$$f'(x) = \frac{x\cdot\dfrac{\cos x}{\sin x} - \log(\sin x)}{x^2}.$$

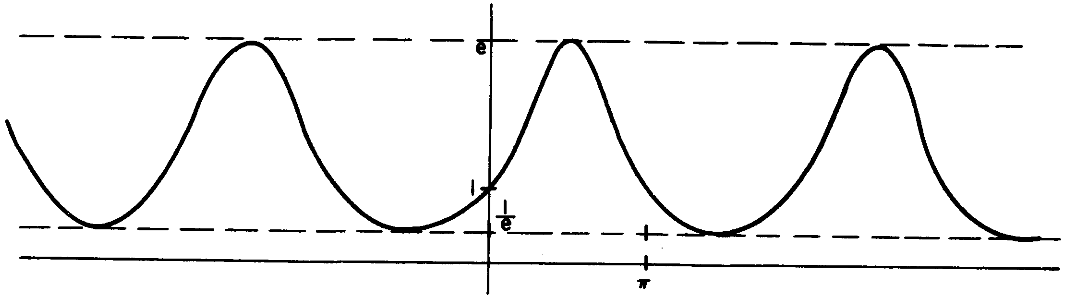
(viii) $4\log(3+e^4)\,e^{4x} + (\log 3)(\arcsin x)^{(\log 3)-1}\dfrac{1}{\sqrt{1-x^2}}.$

(x) $x^x(1+\log x)$.

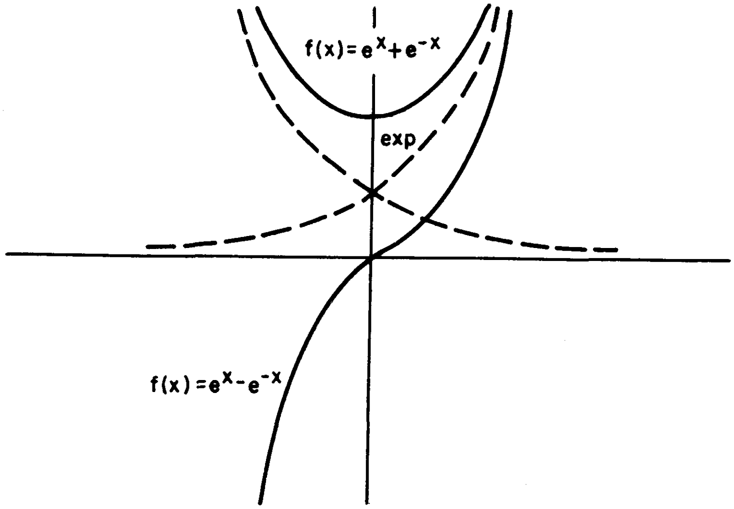
2. (a)



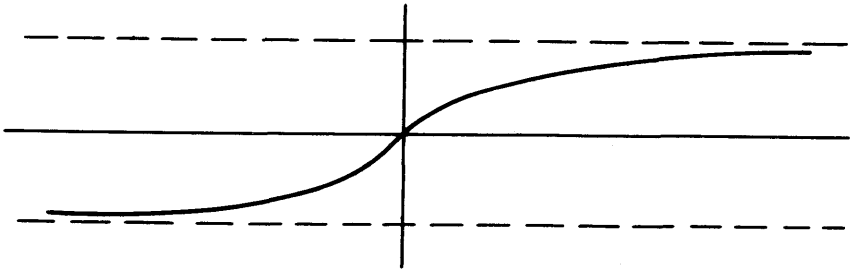
(b)



(c), (d)



(e)



3. (ii)

$$\lim_{x\to 0}\frac{e^x - 1 - x - x^2/2 - x^3/6}{x^3} = \lim_{x\to 0}\frac{e^x - 1 - x - x^2/2}{3x^2}$$

$$= \lim_{x\to 0}\frac{e^x - 1 - x}{6x} = \lim_{x\to 0}\frac{e^x - 1}{6} = 0.$$

(iv)

$$\lim_{x\to 0}\frac{\log(1 + x) - x + x^2/2}{x^2} = \lim_{x\to 0}\frac{\dfrac{1}{1 + x} - 1 + x}{2x}$$

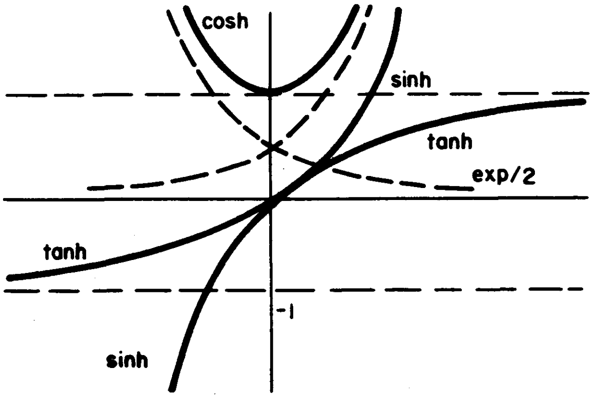
$$= \lim_{x\to 0}\frac{-\dfrac{1}{(1 + x)^2} + 1}{2} = 0.$$

(vi)

$$\lim_{x\to 0}\frac{\log(1 + x) - x + x^2/2 - x^3/3}{x^3} = \lim_{x\to 0}\frac{\dfrac{1}{1 + x} - 1 + x - x^2}{3x^2}$$

$$= \lim_{x\to 0}\frac{\dfrac{-1}{(1 + x)^2} + 1 - 2x}{6x} = \lim_{x\to 0}\frac{\dfrac{2}{(1 + x)^3} - 2}{6} = 0.$$

4.



5. (b) Since $\cosh^2 - \sinh^2 = 1$ by part (a), we have

$$1 - \frac{\sinh^2}{\cosh^2} = \frac{1}{\cosh^2}.$$

(d)

$$\cosh x \cosh y + \sinh x \sinh y = \frac{e^x + e^{-x}}{2}\cdot\frac{e^y + e^{-y}}{2} + \frac{e^x - e^{-x}}{2}\cdot\frac{e^y - e^{-y}}{2}$$

$$= \frac{e^{x+y}}{4} + \frac{e^{-x-y}}{4} + \frac{e^{x-y}}{4} + \frac{e^{y-x}}{4} + \frac{e^{x+y}}{4} + \frac{e^{-x-y}}{4} - \frac{e^{y-x}}{4} - \frac{e^{x-y}}{4}$$

$$= \frac{e^{x+y} + e^{-(x+y)}}{2} = \cosh(x + y).$$

(f) Since

$$\cosh x = \frac{e^x + e^{-x}}{2},$$

we have

$$\cosh'(x) = \frac{e^x - e^{-x}}{2} = \sinh x.$$

6. (b) It follows from Problem 5(a) that

$$\cosh^2(\text{arg sinh } x) = 1 + \sinh^2(\text{arg sinh } x) = 1 + x^2,$$

so

$$\cosh(\text{arg sinh } x) = \sqrt{1 + x^2},$$

since $\cosh y \geq 0$ for all $y$.

(d)

$$(\arg\cosh)'(x) = \frac{1}{\cosh'(\arg\cosh x)} = \frac{1}{\sinh(\arg\cosh x)} = \frac{1}{\sqrt{x^2 - 1}}.$$

8. (b)

$$\lim_{x\to\infty} \frac{x}{(\log x)^n} = \lim_{y\to\infty} \frac{e^y}{y^n} = \infty.$$

(d)

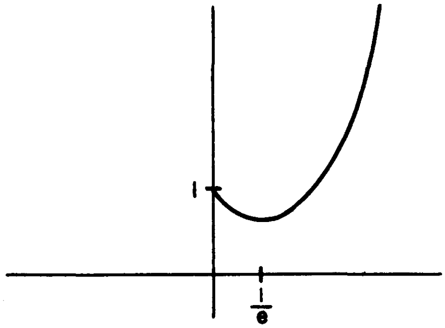
$$\lim_{x\to 0^+} x(\log x)^n = \lim_{x\to 0^+} \frac{(-1)^n\left(\log \frac{1}{x}\right)^n}{\frac{1}{x}}$$

$$= \lim_{y\to\infty} \frac{(-1)^n(\log y)^n}{y} = 0.$$

9. [f is convex, since

$$f'(x) = x^x(1 + \log x),$$

$$f''(x) = x^x(1 + \log x)^2 + \frac{x^x}{x} \geq 0.]$$



10. (a) If $x > 0$ and

$$0 = f'(x) = \frac{x^n e^x - nx^{n-1}e^x}{x^{2n}} = \frac{e^x(x-n)}{x^{n+1}},$$

then $x = n$, so the minimum is at $n$, since $\lim_{x\to 0^+} f(x) = \infty = \lim_{x\to\infty} f(x)$. So for $x > n$ we have $f(x) > f(n) = e^n/n^n$.

(b) If $x > n + 1$, then

$$f'(x) > \frac{e^x}{x^{n+1}} > \frac{e^{n+1}}{(n+1)^{n+1}}$$

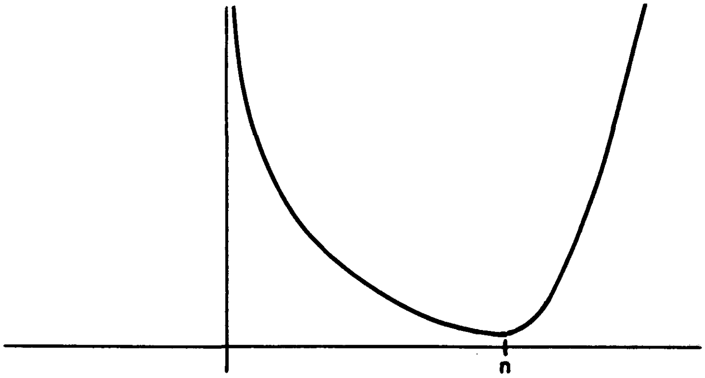
by part (a) applied in the case $n + 1$. It follows immediately that $\lim_{x\to\infty} f(x) = \infty$ (merely using the fact that $f'(x) > \epsilon > 0$ for some $\epsilon$ and all sufficiently large $x$).

11. [f is convex, since

$$f'(x) = e^x x^{-n} - ne^x x^{n-1},$$

$$f''(x) = e^x x^{-n} - ne^x x^{-n-1} - ne^x x^{-n-1} + n(n+1)e^x x^{-n-2}$$

$$= \frac{e^x}{x^{n+2}}[x^2 - 2nx + n^2 + n] \geq 0 \qquad \text{for all } x.]$$



12. (e) If $f(x) = e^{bx}$, then $f'(0) = b$, so

$$\lim_{y \to 0} \frac{e^{by} - 1}{y} = b.$$

Thus

$$\lim_{x \to \infty} x(e^{b/x} - 1) = b.$$

So

$$\begin{aligned} \log b &= \lim_{x \to \infty} x(e^{(\log b)/x} - 1) \\ &= \lim_{x \to \infty} x(b^{1/x} - 1). \end{aligned}$$

13. We have $\lim\limits_{x \to \infty} f(x) = e$ by Problem 12(c) and

$$\begin{aligned} \lim_{x \to 0^+} f(x) &= \lim_{x \to 0^+} \left(1 + \frac{1}{x}\right)^x = \exp(\lim_{x \to 0^+} x \log(1 + \frac{1}{x})) \\ &= \exp(\lim_{x \to 0^+} x \log(\frac{x + 1}{x})) = \exp(\lim_{x \to 0^+} [x \log(x + 1) - x \log x]) \\ &= \exp 0 = 1, \qquad \text{using Problem 8(d).} \end{aligned}$$

Moreover,

$$f'(x) = \left(1 + \frac{1}{x}\right)^x \left[\log(1 + \frac{1}{x}) - \frac{1}{x + 1}\right] = \left(1 + \frac{1}{x}\right)^x g(x).$$

To analyze $f'$, we notice that

$$\begin{aligned} g'(x) &= \frac{1}{1 + \frac{1}{x}} \cdot \frac{-1}{x^2} + \frac{1}{(x + 1)^2} \\ &= \frac{-1}{x(x + 1)^2} < 0. \end{aligned}$$

Thus g is decreasing. Since $\lim\limits_{x\to\infty} g(x) = 0$, we must have $g(x) > 0$ for all $x > 0$. So f is increasing. [Moreover,

$$\begin{aligned}
f''(x) &= \left(1 + \frac{1}{x}\right)^x \left[\log\left(1 + \frac{1}{x}\right) - \frac{1}{x + 1}\right]^2 + \left(1 + \frac{1}{x}\right)^x \left[-\frac{1}{(x + 1)^2}\right] \\
&= \left(1 + \frac{1}{x}\right)^x \left[\left[\log\left(1 + \frac{1}{x}\right)\right]^2 - \frac{2\log\left(1 + \frac{1}{x}\right)}{x + 1} + \frac{1}{(x + 1)^2} - \frac{1}{(x + 1)^2}\right] \\
&= \left(1 + \frac{1}{x}\right)^x \log\left(1 + \frac{1}{x}\right)\left[\log\left(1 + \frac{1}{x}\right) - \frac{2}{x + 1}\right] \\
&= \left(1 + \frac{1}{x}\right)^x \log\left(1 + \frac{1}{x}\right)h(x).
\end{aligned}$$

Now

$$\begin{aligned}
h'(x) &= \frac{1}{1 + \frac{1}{x}} \cdot \frac{-1}{x^2} + \frac{2}{(x + 1)^2} \\
&= \frac{x - 1}{x(x + 1)^2}.
\end{aligned}$$

Thus h is decreasing on $(0, 1]$ and increasing on $[1, \infty)$. But

$$\begin{aligned}
\lim_{x\to 0} h(x) &= \lim_{x\to 0} \log\left(1 + \frac{1}{x}\right) - \frac{2}{x + 1} \\
&= \lim_{x\to 0} \frac{(x + 1)\log\left(\frac{x + 1}{x}\right) - 2}{x + 1} \\
&= \lim_{x\to 0} \frac{(x + 1)\log(x + 1) - (x + 1)\log x - 2}{x + 1} \\
&= -\infty
\end{aligned}$$

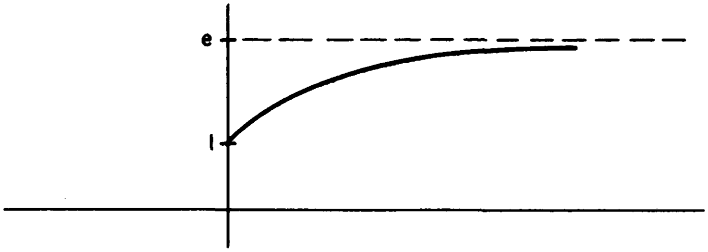
and

$$\lim_{x \to \infty} h(x) = \lim_{x \to \infty} \log(1 + \frac{1}{x}) - \frac{2}{x + 1} = 0$$

and

$$h(1) = \log 2 - 2 < 0,$$

so $h(x) < 0$ for all $x$. Consequently $f''(x) < 0$ for all $x$, so $f$ is concave].



19. Notice that $f$ is continuous, by Theorem 13-8. We therefore have $f'(x) = f(x)$, so there is a number $c$ such that $f(x) = ce^x$. But $f(0) = 0$, so $c = 0$.

20. We have $f''(t) = f'(t)$, so $f'(t) = ce^t$ for some $c$, so $f(t) = a + ce^t$ for some $a$. So

$$\begin{aligned} ce^t &= (a + ce^t) + \int_0^1 (a + ce^t)\, dt \\ &= a + ce^t + a + ce - c, \end{aligned}$$

so $a = c(1 - e)/2$.

21. (a) For $n = 0$ the inequality reads $1 \le e^x$ for $x \ge 0$, which is certainly true, since $e^0 = 1$ and exp is increasing. Suppose the inequality is true for $n$. Let

$$f(x) = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^{n+1}}{(n+1)!}.$$

Then

$$f'(x) = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} \le e^x,$$

while $f(0) = e^0$. It follows that $f(x) \le e^x$ for $x \ge 0$.

(b)

$$\lim_{n\to\infty} \frac{e^x}{x^n} \ge \lim_{n\to\infty} \frac{1 + x + x^2/2! + \dots + x^{n+1}/(n+1)!}{x^n}$$

$$= \lim_{n\to\infty} \frac{1}{x^n} + \frac{1}{x^{n-1}} + \frac{1}{2!x^{n-2}} + \dots + \frac{x}{(n+1)!}$$

$$= \infty.$$

22. Using the form of l'Hopital's Rule which was proved in the answer to Problem 11-38, we have

$$\lim_{x\to\infty} \frac{e^x}{x^n} = \lim_{x\to\infty} \frac{e^x}{nx^{n-1}} = \dots = \lim_{x\to\infty} \frac{e^x}{n!} = \infty.$$

23. If $A(t) = P(t) - 10^7$, then

$$A'(t) = P'(t) = 10^7 - P(t) = -A(t).$$

So (by Problem 16) there is some number $k$ such that

$$A(t) = ke^{-t}.$$

Since $A(0) = P(0) - 10^7 = -10^7$, we obtain $k = -10^7$, so

$$P(t) - 10^7 = -10^7e^{-t},$$

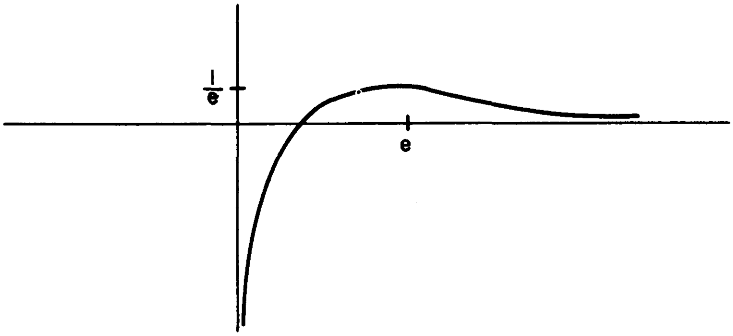
so

$$10^7t = \text{Nap}\log[10^7 - P(t)]$$
$$= \text{Nap}\log 10^7e^{-t};$$

letting $x = 10^7e^{-t}$, so that $t = \log(10^7/x)$, we obtain

$$\text{Nap}\log x = 10^7 \log \frac{10^7}{x}.$$

24. (a) We have $\lim_{x\to 0^+} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = 0$ by Problem 8.



(b) Since f has its maximum at e, we have

$$\frac{\log e}{e} > \frac{\log \pi}{\pi},$$

so

$$e \log \pi > \pi \log e,$$

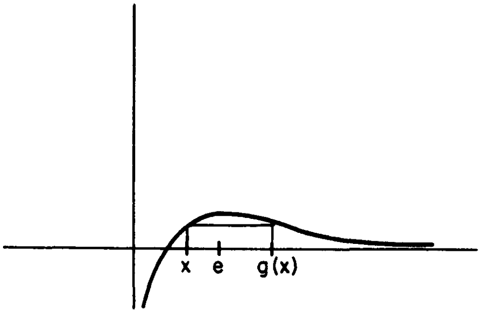
so

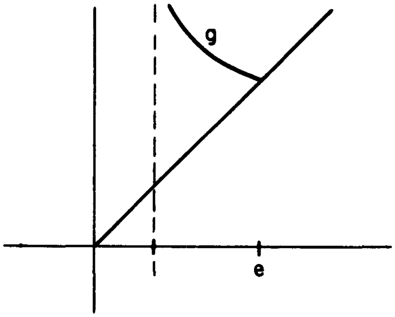
$$\pi^e > e^\pi.$$

(c) The equation $x^y = y^x$ is equivalent to $f(x) = f(y)$. The assertions in part (c) amount to the fact that the values $f(x)$ for $0 < x < 1$ or $x = e$ are taken on only once, while the values $f(x)$ for $1 < x < e$ are taken on for some $x' > e$, and vice versa.

(d) Part (c) shows that the only possible natural numbers $x < y$ with $x^y = y^x$ must involve $1 < x < e$, so $x = 2$.

(e), (f) If $f_1$ and $f_2$ are defined as in part (f), then $g = f_2^{-1} \circ f_1$. The curve in part (e) is the graph of g on (1, e); the straight line is the graph of the identity function. They "intersect" at (e, e) (more precisely $\lim_{x \to e} g(x) = e$).





Moreover, g is differentiable, since $f_1$ and $f_2$ are differentiable and $f_2'(x) \neq 0$ for all x in the domain of $f_2$. In fact, we have

$$g'(x) = (f_2^{-1}\circ f_1)'(x) = (f_2^{-1})'(f_1(x))\cdot f_1'(x)$$

$$= \frac{1}{f_2'(f_2^{-1}(f_1(x)))}\cdot f_1'(x)$$

$$= \frac{[g(x)]^2}{1 - \log g(x)}\cdot\frac{1 - \log x}{x^2}.$$

25. (a) exp is convex, since $\exp''(x) = \exp(x) > 0$ for all $x$. Similarly, log is concave, since $\log''(x) = -1/x^2 < 0$ for all $x > 0$.

(b) Naturally we are assuming that $z_i > 0$. Problem 8 of the Appendix to Chapter 11, applied to the convex function exp, shows that

$$\exp\left(\sum_{i=1}^{n} p_i \log z_i\right) < \sum_{i=1}^{n} \exp(\log z_i)$$

or

$$z_1^{p_1}\cdot\ldots\cdot z_n^{p_n} \leq p_1z_1 + \ldots + p_nz_n.$$

(c) Choose $p_i = 1/n$.

26. From $f' = f$ we conclude that $f(x) = ce^x$ for some $c$. From

$$f(x + 0) = f(x)f(0)$$

we conclude that either $f(x) = 0$ for all $x$, or else that $f(0) = 1$, in which case $c = 1$.

27. Suppose that $f \neq 0$. From $f(x + 0) = f(x)f(0)$ it follows that $f(0) = 1$. Then from

$$1 = f(0) = f(x + (-x)) = f(x)\cdot f(-x)$$

it follows that $f(x) \neq 0$ for all $x$. Moreover $f(x) > 0$ for all $x$, since

$$f(x) = f(x/2 + x/2) = f(x/2)^2.$$

Now if $n$ is a natural number, then

$$f(n) = f(\underbrace{1 + \dots + 1}_{n \text{ times}}) = f(1)^n;$$

moreover,

$$1 = f(0) = f(n + (-n)) = f(n)\cdot f(-n),$$

so

$$f(-n) = \frac{1}{f(n)} = \frac{1}{f(1)^n} = f(1)^{-n}.$$

Similarly,

$$f(1) = f(\underbrace{\tfrac{1}{n} + \dots + \tfrac{1}{n}}_{n \text{ times}}) = f(\tfrac{1}{n})^n,$$

so $f(\frac{1}{n}) = \sqrt[n]{f(1)} = f(1)^{1/n}$. Finally

$$f(\tfrac{m}{n}) = f(\underbrace{\tfrac{1}{n} + \dots + \tfrac{1}{n}}_{m \text{ times}}) = f(\tfrac{1}{n})^m = f(1)^{m/n}.$$

Since $f$ agrees with $g(x) = [f(1)]^x$ for rational $x$, it follows from Problem 8-6 that $f = g$.

28. If $g(x) = f(e^x)$, then

$$g(x+y) = f(e^{x+y}) = f(e^x \cdot e^y) = f(e^x) + f(e^y) = g(x) + g(y).$$

It follows from Problem 8-7 that $g(x) = cx$ for some $c$. If $c = 0$, then $f = 0$. If $c \neq 0$, then

$$f(e) = f(e^1) = g(1) = c,$$

so

$$f(e^x) = f(e)x$$

or

$$f(x) = f(e) \log x \qquad \text{for } x > 0.$$

29. The formulas for $f'(x)$ and $f''(x)$ (for $x \neq 0$) given in the text suggest the following conjecture, which is easy to prove by induction on $k$:

$$f^{(k)}(x) = e^{-1/x^2} \sum_{i=1}^{3k} \frac{a_i}{x^i} \qquad \text{for some numbers } a_1, \dots, a_{3k}.$$

It is then clear that $f^{(k)}(0) = 0$ for all $k$, using the same argument as in the text.

30. The following conjecture is easy to verify:

$$f^{(k)}(x) = e^{-1/x^2} \left[ \sum_{i=1}^{3k} \frac{a_i}{x^i} \sin \frac{1}{x} + \sum_{i=1}^{3k} \frac{b_i}{x^i} \cos \frac{1}{x} \right] \qquad \text{for some numbers } a_1, \dots, a_{3k}, b_1, \dots, b_{3k}.$$

It is then clear that $f^{(k)}(0) = 0$ for all $k$, as in the previous example (note that $|\sin 1/x| \leq 1$ and $|\cos 1/x| \leq 1$ for all $x \neq 0$).

31. (a) If $y(x) = e^{\alpha x}$, then

$$\begin{aligned} a_n y^{(n)}(x) &+ a_{n-1}y^{(n-1)}(x) + \dots + a_1 y'(x) + a_0 y(x) \\ &= a_n\alpha^n e^{\alpha x} + a_{n-1}\alpha^{n-1}e^{\alpha x} + \dots + a_1\alpha e^{\alpha x} + a_0 e^{\alpha x} \\ &= e^{\alpha x}(a_n\alpha^n + a_{n-1}\alpha^{n-1} + \dots + a_1\alpha + a_0) = 0. \end{aligned}$$

(b) If $y(x) = xe^{\alpha x}$, then

$$y^{(\ell)}(x) = \alpha^{\ell} x e^{\alpha x} + \ell\alpha^{\ell-1}e^{\alpha x}.$$

(This formula can be verified by induction, or deduced from Problem 10-15.) So

$$\begin{aligned} a_n y^{(n)}(x) &+ a_{n-1}y^{(n-1)}(x) + \dots + a_1 y'(x) + a_0 y(x) \\ &= xe^{\alpha x}[a_n\alpha^n + a_{n-1}\alpha^{n-1} + \dots + a_1\alpha + a_0] \\ &\quad + e^{\alpha x}[na_n\alpha^{n-1} + \dots + a_1] = 0 \end{aligned}$$

(the second term in brackets is 0 because $\alpha$ is a double root of (*)).

(c) If $y(x) = x^k e^{\alpha x}$, then by Problem 10-15,

$$y^{(\ell)}(x) = \left[ \sum_{s=0}^{k} \binom{\ell}{s} \frac{k!}{(k-s)!} x^{k-s} \alpha^{\ell-s} \right] e^{\alpha x}.$$

So

$$\sum_{\ell=0}^{n} a_\ell y^{(\ell)}(x) = \sum_{s=0}^{k} \left[ \sum_{\ell=0}^{n} \binom{\ell}{s} a_\ell \alpha^{\ell-s} \right] \frac{k!}{(k-s)!} x^{k-s} e^{\alpha x} = 0$$

(the terms in brackets are 0 because $\alpha$ is a root of (*) of order $s + 1$, for each $s \le k$).

(d) If $y_1, \ldots, y_n$ satisfy (**), then

$$\sum_{\ell=0}^{n} a_\ell (c_1 y_1 + \ldots + c_n y_n)^{(\ell)} = \sum_{j=1}^{n} \left( c_j \sum_{\ell=0}^{n} a_\ell y_j^{(\ell)} \right) = 0.$$

32. (a) From

$$0 = f'(f'' - f) = f'f'' - ff' = \tfrac{1}{2}[(f')^2 - f^2]'$$

it follows that $(f')^2 - f^2$ is constant. The constant must be 0, since $f(0) = f'(0) = 0$.

(b) Since $f(x) \neq 0$ for $x$ in $(a, b)$, it follows from part (a) that either $f'(x) = f(x)$ for all $x$ in $(a, b)$ or else $f'(x) = -f(x)$ for all $x$ in $(a, b)$. Thus either $f(x) = ce^x$ or else $f(x) = ce^{-x}$ for all $x$ in $(a, b)$.

(c) Let $a$ be the largest number in $[0, x_0]$ with $f(a) = 0$. Then $f(x) \neq 0$ for $x$ in $(a, x_0)$. But then $f(x) = ce^x$ or $f(x) = ce^{-x}$ for all $x$ in

$(a, x_0)$, where $c \neq 0$. This contradicts $f(a) = 0$, since $f$ is continuous, since $f(a) = 0 \neq \lim_{x \to a} ce^x$ or $\lim_{x \to a} ce^{-x}$.

33. (a) Let

$$a = \frac{f(0) + f'(0)}{2}$$

$$b = \frac{f(0) - f'(0)}{2}.$$

If $g(x) = ae^x + be^{-x} - f(x)$, then $g'' - g = 0$, so $f(x) = ae^x + be^{-x}$.

(b) Note that

$$ae^x + be^{-x} = (a - b)\,\frac{e^x - e^{-x}}{2} + (a + b)\,\frac{e^x + e^{-x}}{2}$$

$$= (a - b) \sinh x + (a + b) \cosh x.$$

(Comparing with part (a) we see that

$$f(x) = f'(0) \sinh x + f(0) \cosh x,$$

in exact analogy with the trigonometric functions.)

34. (a) We have $f^{(n-1)}(x) = ce^x$, so

$$f(x) = a_0 + a_1x + \dots + a_{n-2}x^{n-2} + ce^x.$$

(b) We have

$$f^{(n-2)}(x) = ce^x + de^{-x}$$

by Problem 33, so

$$f(x) = a_0 + a_1x + \dots + a_{n-3}x^{n-3} + ce^x + de^{-x}.$$

35. (a) Since

$$\begin{aligned} g'(x) &= f'(x_0 + x)f(x_0 - x) - f(x_0 + x)f'(x_0 - x) \\ &= f(x_0 + x)f(x_0 - x) - f(x_0 + x)f(x_0 - x) = 0, \end{aligned}$$

the function g is constant. Moreover, $g(0) = f(x_0)^2 \neq 0$. So

$$f(x_0 + x)f(x_0 - x) \neq 0 \quad \text{for all } x,$$

which implies that $f(x) \neq 0$ for all x.

(b) Let $f = f_1/f_1(0)$, where $f_1 \neq 0$ and $f_1' = f_1$.

(c) Since

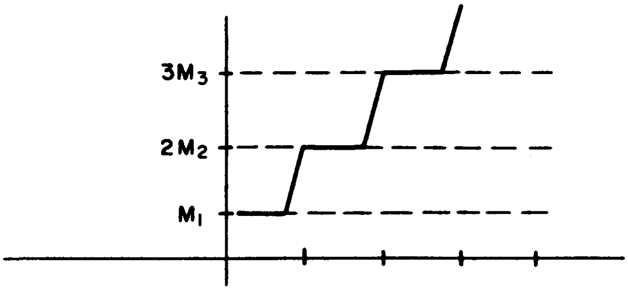
$$\begin{aligned} g'(x) &= \frac{f(x)f'(x + y) - f(x + y)f'(x)}{f(x)^2} \\ &= \frac{f(x)f(x + y) - f(x + y)f(x)}{f(x)^2} = 0, \end{aligned}$$

the function g is constant, and clearly $g(0) = f(y)$, so $f(x + y)/f(x) = f(y)$ for all x.

(d) f is increasing, since $f(x) = f(x/2 + x/2) = [f(x/2)]^2 > 0$. Moreover,

$$\begin{aligned}(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))}\\ &= \frac{1}{f(f^{-1}(x))} = \frac{1}{x}.\end{aligned}$$

36. Let $M_n$ be the maximum of $|g_1| + \dots + |g_n|$ on $[0, n]$ and choose f so that $f(x) \geq nM_n$ on $[0, n]$.



37. If there were natural numbers a and b with $\log_{10} 2 = a/b$, then

$$2 = 10^{a/b},$$

so $2^b = 10^a$. This contradicts the fact, mentioned in Problem 1-16, that an integer can be factored uniquely into primes (since the product $2^b$ does not involve the prime 5, while the product $10^a$ does).

# CHAPTER 18

2. (ii) $-e^{-x^2}/2$. (Let $u = -x^2$.)

(iv) $-1/(e^x + 1)$. (Let $u = e^x$.)

(vi) $(\arcsin x^2)/2$. (Let $u = x^2$.)

(viii) $-(1 - x^2)^{3/2}/3$. (Let $u = 1 - x^2$.)

(x) $[\log(\log x)]^2/2$. (Let $u = \log(\log x)$.)

3. (ii)

$$\int x^3 e^{x^2}\,dx = \int x^2(xe^{x^2})\,dx = \frac{x^2e^{x^2}}{2} - \int xe^{x^2}\,dx$$

$$= \frac{x^2e^{x^2}}{2} - \frac{e^{x^2}}{2}.$$

(iv)

$$\int x^2 \sin x\,dx = x^2(-\cos x) + 2\int x\cos x\,dx$$

$$= -x^2\cos x + 2[x\sin x - \int \sin x\,dx]$$

$$= -x^2\cos x + 2x\sin x + 2\cos x.$$

(vi)

$$\int \log(\log x)\cdot\frac{1}{x}\,dx = (\log x)\cdot\log(\log x) - \int \log x\cdot\frac{1}{\log x}\cdot\frac{1}{x}\,dx$$

$$= (\log x)\cdot\log(\log x) - \log x.$$

(viii)

$$\begin{aligned}\int 1\cdot\cos(\log x)\,dx &= x\cos(\log x) + \int x\sin(\log x)\cdot\frac{1}{x}\,dx\\ &= x\cos(\log x) + \int 1\cdot\sin(\log x)\,dx\\ &= x\cos(\log x) + [x\sin(\log x) - \int x\cos(\log x)\cdot\frac{1}{x}\,dx,\end{aligned}$$

so

$$\int \cos(\log x)\,dx = \frac{x\cos(\log x) + x\sin(\log x)}{2}.$$

(x)

$$\begin{aligned}\int x(\log x)^2\,dx &= \frac{x^2(\log x)^2}{2} - \int \frac{x^2}{2}\cdot 2\log x\cdot\frac{1}{x}\,dx\\ &= \frac{x^2(\log x)^2}{2} - \int x\log x\,dx\\ &= \frac{x^2(\log x)^2}{2} - (\frac{x^2\log x}{2} - \int \frac{x^2}{2}\cdot\frac{1}{x}\,dx)\\ &= \frac{x^2(\log x)^2}{2} - \frac{x^2\log x}{2} + \frac{x^3}{6}.\end{aligned}$$

4. (ii) Let $x = \tan u$, $dx = \sec^2 u\,du$. The integral becomes

$$\begin{aligned}\int \frac{\sec^2 u\,du}{\sqrt{1+\tan^2 u}} &= \int \sec u\,du = \log(\sec u + \tan u)\\ &= \log(x + \sqrt{1+x^2}).\end{aligned}$$

(iv) Let $x = \sec u$, $dx = \sec u\tan u\,du$. The integral becomes

$$\int \frac{\sec u \tan u \, du}{\sec u\sqrt{\sec^2 u - 1}} = \int 1 \, du = u = \text{arcsec } x.$$

(This can be also expressed in terms of more familiar functions as $\arctan(\sqrt{x^2 - 1})$.)

(vi) Let $x = \tan u$, $dx = \sec^2 u \, du$. The integral becomes

$$\int \frac{\sec^2 u \, du}{\tan u\sqrt{1 + \tan^2 u}} = \int \frac{\sec u \, du}{\tan u \, du} = \int \csc u \, du$$

$$= -\log(\csc u + \cot u) = -\log\left(\frac{1}{x} + \frac{\sqrt{1 + x^2}}{x}\right)$$

$$= \log\left(\frac{x}{1 + \sqrt{1 + x^2}}\right).$$

(viii) Let $x = \sin u$, $dx = \cos u \, du$. The integral becomes

$$\int \sqrt{1 - \sin^2 u} \cos u \, du = \int \cos^2 u \, du = \int \frac{1 + \cos 2u}{2} du$$

$$= \frac{u}{2} + \frac{\sin 2u}{4}$$

$$= \frac{u}{2} + \frac{\sin u \cos u}{2}$$

$$= \frac{\arcsin x}{2} + \frac{x\sqrt{1 - x^2}}{2}.$$

(x) Let $x = \sec u$, $dx = \sec u \tan u \, du$. The integral becomes

$$\int \sqrt{\sec^2 u - 1}\ \sec u \tan u\ du = \int \sec u \tan^2 u\ du$$

$$= \int (\sec u)(\sec^2 u - 1)\ du = \int \sec^3 u\ du - \int \sec u\ du$$

$$= \frac{1}{2}[\tan u \sec u + \log(\sec u + \tan u)] \qquad \text{by Problem 3(vii)}$$

$$- \log(\sec u + \tan u)$$

$$= \frac{1}{2}\, x\sqrt{x^2 - 1} - \frac{1}{2}\log(x + \sqrt{x^2 - 1}).$$

5. (ii) Let $u = e^x$, $x = \log u$, $dx = 1/u\ du$. The integral becomes

$$\int \frac{du}{u(1 + u)} = \int \frac{1}{u} - \frac{1}{1 + u}\ du$$

$$= \log u - \log(1 + u)$$

$$= x - \log(1 + e^x).$$

(iv) Let $u = \sqrt{1 + e^x}$, $x = \log(u^2 - 1)$, $dx = 2u/(u^2 - 1)\ du$. The integral becomes

$$\int \frac{2u\ du}{u(u^2 - 1)} = \int \frac{1}{u + 1} + \frac{1}{u - 1}\ du$$

$$= \log(u + 1) + \log(u - 1)$$

$$= \log(1 + \sqrt{1 + e^x}) + \log(\sqrt{1 + e^x} - 1).$$

(vi) Let $u = \sqrt{\sqrt{x} + 1}$, $x = (u^2 - 1)^2$, $dx = 4u(u^2 - 1)\ du$. The integral becomes

$$\int \frac{4u(u^2 - 1)\ du}{u} = \frac{4}{3}u^3 - 4u$$

$$= \frac{4}{3}(\sqrt{x} + 1)^{3/2} - 4(\sqrt{x} + 1)^{1/2}.$$

(viii) Let $u = \sqrt{x}$, $x = u^2$, $dx = 2u\ du$. The integral becomes

$$\int 2ue^u\ du = 2ue^u - 2\int e^u\ du$$

$$= 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}}.$$

(x) Let $u = 1/x$, $x = 1/u$, $dx = -1/u^2\ du$. The integral becomes

$$\int \sqrt{\frac{1/u - 1}{1/u + 1}}\cdot u^2 \cdot -\frac{1}{u^2}\ du = -\int \frac{\sqrt{1 - u}}{\sqrt{1 + u}}\ du = -\int \frac{1 - u}{\sqrt{1 - u^2}}\ du.$$

Now let $u = \sin t$, $du = \cos t\ dt$. The integral becomes

$$\int \frac{1 - \sin t}{\cos t} \cos t\ dt = \int 1 - \sin t\ dt$$

$$= t + \cos t$$

$$= \arcsin u + \sqrt{1 - u^2}$$

$$= \arcsin \frac{1}{x} + \frac{\sqrt{x^2 - 1}}{x}.$$

6. In this answer set, I will denote the original integral.

(ii)

$$I = \int \frac{2}{(x - 1)^2}\ dx + \frac{3}{(x - 1)^3}\ dx = -\frac{2}{(x - 1)} - \frac{3}{2(x - 1)^2}.$$

(iv)

$$I = \int \frac{1}{x + 3} + \frac{1}{x - 1} + \frac{1}{(x - 1)^2}\,dx = \log(x + 3) + \log(x - 1) - \frac{1}{x - 1}.$$

(vi)

$$I = \int \frac{x}{x^2 + 1} + \frac{2}{(x^2 + 1)^2}\,dx$$

$$= \frac{\log(x^2 + 1)}{2} + 2\left[\frac{x}{2(x^2 + 1)} + \frac{1}{2}\int \frac{1}{x^2 + 1}\,dx\right]$$

$$= \frac{\log(x^2 + 1)}{2} + \frac{x}{x^2 + 1} + \arctan x.$$

(viii)

$$I = \int \frac{dx}{x^4 + 2x^2 + 1 - 2x^2} = \int \frac{dx}{(x^2 + 1)^2 - 2x^2}$$

$$= \int \frac{dx}{(x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)} = \int \frac{\frac{\sqrt{2}}{4}x + \frac{1}{2}}{(x^2 + \sqrt{2}x + 1)} + \frac{-\frac{\sqrt{2}}{4}x + \frac{1}{2}}{(x^2 - \sqrt{2}x + 1)}\,dx$$

$$= \frac{\sqrt{2}}{8}\int \frac{(2x + \sqrt{2})\,dx}{(x^2 + \sqrt{2}x + 1)} + \frac{1}{4}\int \frac{dx}{(x^2 + \sqrt{2}x + 1)}$$

$$- \frac{\sqrt{2}}{8}\int \frac{(2x - \sqrt{2})\,dx}{(x^2 - \sqrt{2}x + 1)} + \frac{1}{4}\int \frac{dx}{(x^2 - \sqrt{2}x + 1)}$$

$$= \frac{\sqrt{2}}{8}\log(x^2 + \sqrt{2}x + 1) - \frac{\sqrt{2}}{8}\log(x^2 - \sqrt{2}x + 1)$$

$$+ \frac{1}{2}\int \frac{dx}{(\sqrt{2}x + 1)^2 + 1} + \frac{1}{2}\int \frac{dx}{(-\sqrt{2}x + 1)^2 + 1}$$

$$= \frac{\sqrt{2}}{8}\log(x^2 + \sqrt{2}x + 1) - \frac{\sqrt{2}}{8}\log(x^2 - \sqrt{2}x + 1)$$

$$+ \frac{\sqrt{2}}{4}\arctan(\sqrt{2}x + 1) - \frac{\sqrt{2}}{4}\arctan(-\sqrt{2}x + 1).$$

(x)

$$I = \frac{3}{2}\int \frac{2x + 1}{(x^2 + x + 1)^3}\,dx - \frac{3}{2}\int \frac{1}{(x^2 + x + 1)^3}\,dx$$

$$= \frac{-3}{4(x^2 + x + 1)^2} - \frac{3}{2}\int \frac{dx}{\left((x + \frac{1}{2})^2 + \frac{3}{4}\right)^3}$$

$$= \frac{-3}{4(x^2 + x + 1)^2} - \frac{3}{2}\left(\frac{4}{3}\right)^3 \int \frac{dx}{\left(\left(\frac{x + \frac{1}{2}}{\sqrt{3/4}}\right)^2 + 1\right)^3} \quad (\text{let } u = \frac{x + \frac{1}{2}}{\sqrt{3/4}},\ dx = \sqrt{3/4}\,du)$$

$$= \frac{-3}{4(x^2 + x + 1)^2} - 32\sqrt{3/4}\int \frac{du}{(u^2 + 1)^3}$$

$$= \frac{-3}{4(x^2 + x + 1)^2} - 32\sqrt{3/4}\,[\frac{1}{4}\frac{u}{(u^2 + 1)^2} + \frac{3}{4}\int \frac{du}{(u^2 + 1)^2}]$$

$$= \frac{-3}{4(x^2 + x + 1)^2} - \frac{8\sqrt{3/4}\,u}{(u^2 + 1)^2} - 24\sqrt{3/4}\,[\frac{1}{2}\frac{u}{(u^2 + 1)} + \frac{1}{2}\int \frac{1}{u^2 + 1}\,du]$$

$$= \frac{-3}{4(x^2 + x + 1)^2} - \frac{8x + 4}{(x^2 + x + 1)}\cdot\frac{4}{3} - \frac{12x + 6}{x^2 + x + 1}\sqrt{\frac{4}{3}} + \frac{12}{x^2 + x + 1}.$$

7. (i) $(\arctan x)^2/2$.

(ii)

$$\int \frac{x}{(1+x^2)^3} \arctan x \, dx = \frac{-1}{4(1+x^2)^2} \arctan x - \int \frac{x \, dx}{(1+x^2)^4}$$

$$= \frac{-\arctan x}{4(1+x^2)^2} + \frac{1}{6(1+x^2)^3}.$$

(iii)

$$\int \log\sqrt{1+x^2} \, dx = x \log\sqrt{1+x^2} - \int x \cdot \frac{1}{\sqrt{1+x^2}} \cdot \frac{2x}{\sqrt{1+x^2}} \, dx$$

$$= x \log\sqrt{1+x^2} - \int \frac{2x^2}{1+x^2} \, dx$$

$$= x \log\sqrt{1+x^2} - \int 2 + \frac{-2}{1+x^2} \, dx$$

$$= x \log\sqrt{1+x^2} - 2x + 2 \arctan x.$$

(iv)

$$\int x \log\sqrt{1+x^2} \, dx = \frac{x^2}{2} \log\sqrt{1+x^2} - \int \frac{x^2}{2} \cdot \frac{2x}{1+x^2} \, dx$$

$$= \frac{x^2}{2} \log\sqrt{1+x^2} - \int x - \frac{x}{1+x^2} \, dx$$

$$= \frac{x^2}{2} \log\sqrt{1+x^2} - \frac{x^2}{2} + \frac{\log(1+x^2)}{2}.$$

(v) Let

$$y = \frac{x^2 - 1}{x^2 + 1}, \qquad yx^2 + y = x^2 - 1,$$

$$y + 1 = x^2(1 - y)$$

$$x = \sqrt{\frac{1 + y}{1 - y}},$$

$$dx = \sqrt{\frac{1 - y}{1 + y}}\ \frac{1}{(1 - y)^2}\ dy.$$

The integral becomes

$$\int y \frac{1}{\sqrt{1 + \left(\frac{1 + y}{1 - y}\right)^2}} \frac{\sqrt{1 - y}}{\sqrt{1 + y}} \frac{1}{(1 - y)^2}\ dy$$

$$= \int \frac{y}{(1 - y)^2} \frac{1}{\sqrt{\frac{2 + 2y^2}{(1 - y)^2}}} \frac{\sqrt{1 - y}}{\sqrt{1 + y}}\ dy$$

$$= \int \frac{y}{(1 - y)} \frac{1}{\sqrt{2 + 2y^2}} \frac{\sqrt{1 - y}}{\sqrt{1 + y}}\ dy$$

$$= \frac{1}{\sqrt{2}} \int \frac{y\ dy}{\sqrt{1 - y^2}\sqrt{1 + y^2}}$$ (multiplying top and bottom of the previous expression by $\sqrt{1 - y}$).

Now let $u = y^2$, $du = 2y\ dy$. The integral becomes

$$\frac{1}{2\sqrt{2}} \int \frac{du}{\sqrt{1 - u}\ \sqrt{1 + u}} = \frac{1}{2\sqrt{2}} \int \frac{du}{\sqrt{1 - u^2}}$$

$$= \frac{1}{2\sqrt{2}} \arcsin u$$

$$= \frac{1}{2\sqrt{2}} \arcsin\left(\frac{x^2 - 1}{x^2 + 1}\right)^2.$$

(vi) Let $u = \sqrt{x}$, $x = u^2$, $dx = 2u\,du$. The integral becomes

$$\int 2u \arcsin u\, du = u^2 \arcsin u - \int \frac{u^2}{\sqrt{1 - u^2}}\, du.$$

Now let $u = \sin t$, $du = \cos t\, dt$. The integral becomes

$$\int \frac{\sin^2 t \cos t}{\sqrt{1 - \sin^2 t}}\, dt = \int \sin^2 t\, dt = \int \frac{1 - \cos 2t}{2}\, dt$$

$$= \frac{t}{2} - \frac{\sin 2t}{4}$$

$$= \frac{t}{2} - \frac{\sin t \cos t}{2}.$$

So the original integral is

$$u^2 \arcsin u - \frac{\arcsin u}{2} - \frac{u\sqrt{1 - u^2}}{2}$$

$$= x \arcsin \sqrt{x} - \frac{\arcsin \sqrt{x}}{2} - \frac{\sqrt{x}\,\sqrt{1 - x}}{2}.$$

(vii) Since

$$\int \frac{1}{1 + \sin x}\, dx = \tan x - \sec x \qquad \text{(Problem 1(viii))}$$

we have

$$\int x \cdot \frac{1}{1 + \sin x}\, dx = x(\tan x - \sec x) - \int \tan x - \sec x\, dx$$

$$= x(\tan x - \sec x) + \log(\cos x) + \log(\sec x + \tan x).$$

(viii)

$$\int x \cos x\, e^{\sin x}\, dx - \int \sec x \tan x\, e^{\sin x}\, dx$$

$$= (xe^{\sin x} - \int e^{\sin x}\,dx) - (\sec x\, e^{\sin x} - \int \sec x \cos x\, e^{\sin x}\,dx)$$

$$= xe^{\sin x} - \sec x\, e^{\sin x}.$$

(ix) Let

$$u = \sqrt{\tan x}, \qquad u^2 = \tan x$$

$$x = \arctan u^2$$

$$dx = \frac{2u}{1+u^4}\,du.$$

The integral becomes (compare with Problem 6(viii))

$$\int \frac{2u^2\,du}{1+u^4} = \int \frac{-\frac{\sqrt{2}}{2}u}{u^2+\sqrt{2}u+1} + \frac{\frac{\sqrt{2}}{2}u}{u^2-\sqrt{2}u+1}\,du$$

$$= \frac{-\sqrt{2}}{4}\log(u^2+\sqrt{2}u+1) + \frac{\sqrt{2}}{4}\log(u^2-\sqrt{2}u+1)$$

$$+ \frac{\sqrt{2}}{2}\arctan(\sqrt{2}u+1) - \frac{\sqrt{2}}{2}\arctan(-\sqrt{2}u+1)$$

$$= \frac{-\sqrt{2}}{4}\log(\tan x + \sqrt{2\tan x}+1) + \frac{\sqrt{2}}{4}\log(\tan x - \sqrt{2\tan x}+1)$$

$$+ \frac{\sqrt{2}}{2}\arctan(\sqrt{2\tan x}+1) - \frac{\sqrt{2}}{2}\arctan(-\sqrt{2\tan x}+1).$$

(x)

$$\int \frac{dx}{x^6+1} = \int \frac{dx}{(x^2+1)(x^4-x^2+1)} = \int \frac{dx}{(x^2+1)[(x^4+2x^2+1)-3x^2]}$$

$$= \int \frac{dx}{(x^2+1)[(x^2+1)^2-3x^2]} = \int \frac{dx}{(x^2+1)(x^2+\sqrt{3}x+1)(x^2-\sqrt{3}x+1)}$$

$$= \int \frac{\frac{1}{3}}{x^2 + 1} + \frac{\frac{\sqrt{3}}{6}x + \frac{1}{3}}{x^2 + \sqrt{3}x + 1} + \frac{-\frac{\sqrt{3}}{6}x + \frac{1}{3}}{x^2 - \sqrt{3}x + 1}\,dx$$

$$= \frac{\arctan x}{3} + \frac{\sqrt{3}}{12}\int \frac{2x + \sqrt{3}}{x^2 + \sqrt{3}x + 1}\,dx + \frac{1}{12}\int \frac{dx}{x^2 + \sqrt{3}x + 1}$$

$$- \frac{\sqrt{3}}{12}\int \frac{2x - \sqrt{3}}{x^2 - \sqrt{3}x + 1}\,dx + \frac{1}{12}\int \frac{dx}{x^2 - \sqrt{3}x + 1}$$

$$= \frac{\arctan x}{3} + \frac{\sqrt{3}}{12}\log(x^2 + \sqrt{3}x + 1) - \frac{\sqrt{3}}{12}\log(x^2 - \sqrt{3}x + 1)$$

$$+ \frac{1}{6}\arctan(2x + \sqrt{3}) + \frac{1}{6}\arctan(2x - \sqrt{3}).$$

8. (iv) Let $x = \cosh u$, $dx = \sinh u\,du$. The integral in Problem 4(iv) becomes

$$\int \frac{\sinh u\,du}{\cosh u \sinh u} = \int \frac{1}{\cosh u}\,du$$

$$= \int \frac{2}{e^u + e^{-u}}\,du = \int \frac{2e^u}{1 + e^{2u}}\,du$$

$$= 2\arctan e^u$$

$$= 2\arctan(x + \sqrt{x^2 - 1}),$$

since $u = \arg\cosh x = \log(x + \sqrt{x^2 - 1})$, as found in Problem 17-7.

Comparing with Problem 4(iv) we cannot conclude that

$$2\arctan(x + \sqrt{x^2 - 1}) = \arctan(\sqrt{x^2 - 1}),$$

but only that these two expressions differ by a constant. As a matter of fact, we can only conclude that there are two constants $c_1$ and $c_2$ with

$$2\arctan(x + \sqrt{x^2 - 1}) = \arctan(\sqrt{x^2 - 1}) + c_1 \quad \text{for } x \geq 1,$$

$$2\arctan(x + \sqrt{x^2 - 1}) = \arctan(\sqrt{x^2 - 1}) + c_2 \quad \text{for } x \leq -1.$$

By setting x = 1 and -1 it is easy to see that $c_1 = \pi/2$ and $c_2 = -\pi/2$.

(vi) Let x = sinh u, dx = cosh u du. The integral becomes

$$\int \frac{\cosh u\, du}{\sinh u \cosh u} = \int \frac{du}{\sinh u}$$

$$= \int \frac{2}{e^u - e^{-u}}\, du = \int \frac{2e^u\, du}{e^{2u} - 1}$$

$$= \int \frac{-e^u}{e^u + 1} + \frac{e^u}{e^u - 1}\, du$$

$$= -\log(e^u + 1) + \log(e^u - 1)$$

$$= \log\left(\frac{e^u - 1}{e^u + 1}\right) = \log\left(\frac{\sqrt{x^2 + 1} + x - 1}{\sqrt{x^2 + 1} + x + 1}\right).$$

(ix) Let x = sinh u, dx = cosh u du. The integral becomes

$$\int \cosh^2 u\, du = \int \frac{e^{2u}}{4} + \frac{1}{2} + \frac{e^{-2u}}{4}\, du$$

$$= \frac{e^{2u}}{8} + \frac{u}{2} - \frac{e^{-2u}}{8} = \frac{(x + \sqrt{1 + x^2})^2}{8} + \frac{\log(x + \sqrt{1 + x^2})}{2} - \frac{1}{8(x + \sqrt{1 + x^2})^2}.$$

(x) Let x = cosh u, dx = sinh u du. The integral becomes

$$\int \sinh^2 u\, du = \int \frac{e^{2u}}{4} - \frac{1}{2} + \frac{e^{-2u}}{4}\, du$$

$$= \frac{e^{2u}}{8} - \frac{u}{2} - \frac{e^{-2u}}{8} = \frac{(x + \sqrt{x^2 - 1})^2}{8} - \frac{\log(x + \sqrt{x^2 - 1})}{2} - \frac{1}{8(x + \sqrt{x^2 - 1})^2}.$$

9. (i)

$$\int \frac{1}{1 + \frac{2t}{1 + t^2}} \cdot \frac{2}{1 + t^2}\, dt = \int \frac{2\, dt}{1 + 2t + t^2} = \int \frac{2\, dt}{(1 + t)^2} = \frac{-2}{1 + t} = \frac{-2}{1 + \tan \frac{x}{2}}.$$

Comparing with the formula

$$\int \frac{1}{1 + \sin x}\, dx = \int \frac{1 - \sin x}{1 - \sin^2 x}\, dx = \int \sec^2 x - \sec x \tan x\, dx = \tan x - \sec x,$$

we can conclude that

$$\frac{-2}{1 + \tan \frac{x}{2}} = \tan x - \sec x - 1.$$

(This can be checked most easily by expressing everything in terms of t:

$$\frac{-2}{1 + t} = \frac{2t}{1 - t^2} - \frac{1 + t^2}{1 - t^2} - 1.)$$

(ii) Let $t = \tan x$, $dx = 1/(1 + t^2)\, dt$. Then $\sin^2 x$ can be expressed in terms of t as

$$\sin^2 x = 1 - \cos^2 x = 1 - \frac{1}{\sec^2 x}$$

$$= 1 - \frac{1}{\tan^2 x + 1}$$

$$= 1 - \frac{1}{1 + t^2} = \frac{t^2}{1 + t^2}.$$

So the integral becomes

$$\int \frac{1}{1+\frac{t^2}{1+t^2}}\cdot\frac{1}{1+t^2}\,dt$$

$$= \int \frac{1}{2t^2+1}\,dt = \frac{\arctan\ \sqrt{2}\,t}{\sqrt{2}}$$

$$= \frac{\arctan(\sqrt{2}\tan x)}{\sqrt{2}}.$$

(iii)

$$\int \frac{1}{\frac{2at}{1+t^2}+\frac{b-bt^2}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt = \int \frac{2}{2at+b-bt^2}\,dt.$$

If b > 0, this can be written

$$\int \frac{-2\,dt}{bt^2-2at-b} = \int \frac{-2\,dt}{\left(\sqrt{b}t-\frac{a}{\sqrt{b}}\right)^2-\frac{(a^2+b^2)}{b}}$$

$$= \int \left[\frac{\frac{\sqrt{b}}{\sqrt{a^2+b^2}}}{\left(\sqrt{b}t-\frac{a}{\sqrt{b}}+\frac{\sqrt{a^2+b^2}}{\sqrt{b}}\right)}-\frac{\frac{\sqrt{b}}{\sqrt{a^2+b^2}}}{\left(\sqrt{b}t-\frac{a}{\sqrt{b}}-\frac{\sqrt{a^2+b^2}}{\sqrt{b}}\right)}\right]dt$$

$$= \frac{1}{\sqrt{a^2+b^2}}\left[\log\left(\sqrt{b}t-\frac{a}{\sqrt{b}}+\frac{\sqrt{a^2+b^2}}{\sqrt{b}}\right)-\log\left(\sqrt{b}t-\frac{a}{\sqrt{b}}-\frac{\sqrt{a^2+b^2}}{\sqrt{b}}\right)\right].$$

If $b < 0$, the integral can be written

$$\int \frac{2\,dt}{-bt^2 + 2at + b} = \int \frac{2\,dt}{\left(\sqrt{-b}\,t + \frac{a}{\sqrt{-b}}\right)^2 + \frac{a^2 + b^2}{b}}$$

$$= \frac{1}{\sqrt{a^2 + b^2}}\left[\log\left(\sqrt{-b}\,t + \frac{a}{\sqrt{-b}} - \frac{\sqrt{a^2 + b^2}}{\sqrt{-b}}\right) - \log\left(\sqrt{-b}\,t + \frac{a}{\sqrt{-b}} + \frac{\sqrt{a^2 + b^2}}{\sqrt{b}}\right)\right].$$

It is also possible to write

$$\int \frac{dx}{a \sin x + b \cos x} = \int \frac{dx}{A \sin(x + B)}$$

$$= -\frac{1}{A} \log(\csc(x + B) + \cot(x + B)),$$

where

$$A = \sqrt{a^2 + b^2}$$

$$\sin B = \frac{b}{\sqrt{a^2 + b^2}}$$

$$\cos B = \frac{a}{\sqrt{a^2 + b^2}}.$$

(iv)

$$\int \frac{4t^2}{(1 + t^2)^2} \cdot \frac{2}{1 + t^2}\,dt = 8 \int \frac{1}{(1 + t^2)^2} - \frac{1}{(1 + t^2)^3}\,dt$$

$$\begin{aligned}
&= -8\left[\frac{1}{4}\,\frac{t}{(1+t^2)^2} + \frac{3}{4}\int \frac{1}{(1+t^2)^2}\,dt\right] + 8\int \frac{1}{(1+t^2)^2}\,dt \\
&= \frac{-2t}{(1+t^2)^2} + 2\left[\frac{1}{2}\,\frac{t}{1+t^2} + \frac{1}{2}\int \frac{1}{1+t^2}\,dt\right] \\
&= \frac{-2t}{(1+t^2)^2} + \frac{t}{1+t^2} + \arctan t \\
&= \frac{-2\tan x/2}{\sec^4 x/2} - \frac{\tan x/2}{\sec^2 x/2} + \frac{x}{2} \\
&= -2\sin x/2\cos^3 x/2 - \sin x/2\cos x/2 + x/2 \\
&= -2\sin x/2\cos x/2(1-\sin^2 x/2) - \sin x/2\cos x/2 + x/2 \\
&= -\sin x\left[1 - \left(\frac{1-\cos x}{2}\right)\right] - \frac{\sin x}{2} + \frac{x}{2} \\
&= -\sin x\left[\frac{1+\cos x}{2} - \frac{1}{2}\right] + \frac{x}{2} \\
&= \frac{-\sin 2x}{4} + \frac{x}{2}.
\end{aligned}$$

(v)

$$\begin{aligned}
\int \frac{1}{3+\dfrac{10t}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt &= \int \frac{2\,dt}{3t^2+10t+3} \\
&= \int \frac{3/4}{3t+1} + \frac{-1/4}{t+3}\,dt \\
&= \frac{1}{4}\log(3t+1) - \frac{1}{4}\log(t+3) \\
&= \frac{1}{4}\log\left(3\tan\frac{x}{2}+1\right) - \frac{1}{4}\log\left(\tan\frac{x}{2}+3\right).
\end{aligned}$$

10. (a) The given formula shows that

$$\begin{aligned}\int \sec x\,dx &= \frac{1}{2}\int \frac{\cos x}{1+\sin x}\,dx + \frac{1}{2}\int \frac{\cos x}{1-\sin x}\,dx\\ &= \frac{1}{2}\log(1+\sin x) - \frac{1}{2}\log(1-\sin x)\\ &= \log\sqrt{1+\sin x} - \log\sqrt{1-\sin x}\\ &= \log\sqrt{\frac{1+\sin x}{1-\sin x}}\\ &= \log\sqrt{\frac{(1+\sin x)^2}{1-\sin^2 x}}\\ &= \log\left(\frac{1+\sin x}{\cos x}\right)\\ &= \log(\sec x + \tan x).\end{aligned}$$

(b) With the substitution $t = \tan x/2$ the integral $\int \sec x\,dx$ becomes

$$\begin{aligned}&\int \frac{1+t^2}{1-t^2}\cdot\frac{2}{1+t^2}\,dt\\ &\quad= \int \frac{1}{1+t} + \frac{1}{1-t}\,dt\\ &\quad= \log(1+t) - \log(1-t)\\ &\quad= \log\left(\frac{1+t}{1-t}\right).\end{aligned}$$

Now

$$\sec x + \tan x = \frac{1}{\cos 2(x/2)} + \tan 2(x/2)$$

$$= \frac{1}{\cos^2 x/2 - \sin^2 x/2} + \frac{2 \tan x/2}{1 - \tan^2 x/2}$$

$$= \frac{1}{2 \cos^2 x/2 - 1} + \frac{2 \tan x/2}{1 - \tan^2 x/2}$$

$$= \frac{1}{\dfrac{2}{1 + \tan^2 x/2} - 1} + \frac{2 \tan x/2}{1 - \tan^2 x/2}$$

$$= \frac{1 + \tan^2 x/2 + 2 \tan x/2}{1 - \tan^2 x/2}$$

$$= \frac{1 + 2t + t^2}{1 - t^2} = \frac{(1 + t)^2}{1 - t^2} = \frac{1 + t}{1 - t}.$$

12. (b) If $F = \int f(x)\, dx$, then

$$\int f^{-1}(x)\, dx = \int 1 \cdot f^{-1}(x)\, dx = xf^{-1}(x) - \int x(f^{-1})'(x)\, dx$$

$$= xf^{-1}(x) - \int \frac{x}{f'(f^{-1}(x))}\, dx.$$

The substitution $u = f^{-1}(x)$, $x = f(u)$, $dx = f'(u)\, du$ changes the new integral to

$$\int \frac{f(u)}{f'(u)} \cdot f'(u)\, du = F(u) = F(f^{-1}(x)),$$

so

$$\int f^{-1}(x)\,dx = xf^{-1}(x) - F(f^{-1}(x)).$$

14.

$$\begin{aligned}\int \log(\log x)\,dx &= \int 1\cdot\log(\log x)\,dx\\ &= x\log(\log x) - \int x\cdot\frac{1}{\log x}\cdot\frac{1}{x}\,dx\\ &= x\log(\log x) - \int \frac{1}{\log x}\,dx.\end{aligned}$$

15.

$$\begin{aligned}\int x^2e^{-x^2}\,dx &= \int x(xe^{-x^2})\,dx\\ &= \frac{-e^{-x^2}}{2}x + \frac{1}{2}\int e^{-x^2}\,dx.\end{aligned}$$

16. (Use the substitution $u = e^x$.) The function $g(x) = 1/(x^5 + x + 1)$ has an elementary primitive $G$, since it is a rational function. Then $G\circ\exp$ is a primitive of $f$.

18. (a)

$$\int x^ne^x\,dx = x^ne^x - n\int x^{n-1}e^x\,dx.$$

(b)

$$\begin{aligned}\int (\log x)^n\,dx &= x(\log x)^n - n\int x(\log x)^{n-1}\frac{1}{x}\,dx\\ &= x(\log x)^n - n\int(\log x)^{n-1}\,dx.\end{aligned}$$

19. By Problem 4(x),

$$\int_1^{\cosh x} \sqrt{t^2 - 1}\, dt = \frac{1}{2}\cosh x\sqrt{\cosh^2 x - 1} - \frac{1}{2}\log(\cosh x + \sqrt{\cosh^2 x - 1})$$

$$= \frac{\cosh x \sinh x}{2} - \frac{1}{2}\log(\cosh x + \sinh x)$$

$$= \frac{\cosh x \sinh x}{2} - \frac{x}{2}.$$

20. By Theorem 2, with $g(x) = a + b - x$,

$$\int_a^b f(a + b - x)\, dx = -\int_a^b f(g(x))\cdot g'(x)\, dx$$

$$= -\int_{g(a)}^{g(b)} f(x)\, dx = -\int_b^a f(x)\, dx = \int_a^b f(x)\, dx.$$

21. By Theorem 2, with $g(x) = x/r$,

$$\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx = r^2 \int_{-r}^{r} \frac{1}{r}\sqrt{1 - \left(\frac{x}{r}\right)^2}\, dx$$

$$= r^2 \int_{-r}^{r} \sqrt{1 - [g(x)]^2}\, g'(x)\, dx$$

$$= r^2 \int_{-1}^{1} \sqrt{1 - x^2}\, dx$$

$$= \frac{\pi r^2}{2}.$$

22. The formula is true for $n = 1$ by Problem 14-5. Suppose that it is true for $n$. Let $F(u) = \int_0^u f(t)\, dt$. Then $F$ is a primitive of $f$ with $F(0) = 0$. So

$$\int_0^x \frac{f(u)(x-u)^{n+1}}{(n+1)!}\,du = \frac{F(u)(x-u)^{n+1}}{(n+1)!}\Bigg|_{u=0}^{u=x} - \int_0^x \frac{-F(u)(x-u)^n}{n!}\,dx$$

$$= 0 + \int_0^x \left(\int_0^{u_n} \left(\dots\left(\int_0^{u_1} F(t)\,dt\right) du_1\right)\dots\right) du_n$$

$$= \int_0^x \left(\int_0^{u_n} \left(\dots\left(\int_0^{u_1} \left(\int_0^{u} f(t)\,dt\right) du_1\right)\dots\right)\right) du_n,$$

which can also be written as

$$\int_0^x \left(\int_0^{u_{n+1}} \left(\dots\left(\int_0^{u_1} f(t)\,dt\right) du_1\right)\dots\right) du_{n+1}.$$

23.

$$\lim_{\lambda\to\infty} \int_a^b f(t)\sin\lambda t\,dt = \lim_{\lambda\to\infty}\left[\frac{-f(t)\cos\lambda t}{\lambda}\Bigg|_a^b + \int_a^b \frac{f'(t)\cos\lambda t}{\lambda}\,dt\right]$$

$$= 0,$$

since

$$\left|\frac{-f(t)\cos\lambda t}{\lambda}\Bigg|_a^b\right| \le \frac{1}{\lambda}\left(|f(b)| + |f(a)|\right),$$

$$\left|\int_a^b \frac{f'(t)\cos\lambda t}{\lambda}\,dt\right| \le \frac{1}{\lambda}\int_a^b |f'(t)|\,dt.$$

24. For each $N$ we have

$$\int_a^N u'(x)v(x)\,dx = u(x)v(x)\Bigg|_a^N - \int_a^N u(x)v'(x)\,dx.$$

The desired equation follows by taking limits (and shows that if any two of three symbols involved exist, the third does also).

25. (a) The integral

$$\int_1^\infty e^{-t}t^{x-1}\,dt$$

certainly exists, because $\int_1^\infty t^{-2}\,dt$ exists (Problem 14-15), and for sufficiently large t we have $e^{-t}t^{x-1} < t^{-2}$ (by Theorem 17-6). On the other hand, if $t > 0$, then $e^{-t}t^{x-1} < t^{x-1}$; since the integral $\int_0^1 t^{x-1}\,dt$ exists for $x > 0$ (Problem 14-17), it follows that $\int_0^1 e^{-t}t^{x-1}\,dt$ exists for $x > 0$ (it is an improper integral if $x < 1$).

(b)

$$\begin{aligned}\Gamma(x+1) &= \int_0^\infty e^{-t}t^{x}\,dt\\ &= -e^{-t}t^{x}\Big|_{t=0}^{t=\infty} + \int_0^\infty xe^{-t}t^{x-1}\,dt\\ &= 0 + x\int_0^\infty e^{-t}t^{x-1}\,dt = x\Gamma(x).\end{aligned}$$

(If $x < 1$, then we are also using a second version of integration by parts to take care of the integral from 0 to 1.)

(c)

$$\Gamma(1) = \int_0^\infty e^{-t}\,dt = -e^{-t}\Big|_0^\infty = 1.$$

This proves that $\Gamma(1) = (1-1)!$. If $\Gamma(n) = (n-1)!$, then $\Gamma(n+1) = n\Gamma(n) = n\cdot(n-1)! = n!$, so the formula is true for all n, by induction.

26. (a)

$$\int_0^{\pi/2} \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x \cos x\Big|_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$

$$= \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx.$$

(b)

$$\int_0^{\pi/2} \sin^{2n+1} x\,dx = \frac{2n}{2n+1}\cdot\int_0^{\pi/2} \sin^{2n-1} x\,dx$$

$$= \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\cdot\int_0^{\pi/2} \sin^{2n-3} x\,dx$$

$$= \dots = \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\cdot\ldots\cdot\frac{2}{3}\int_0^{\pi/2} \sin^1 x\,dx$$

$$= \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\cdot\ldots\cdot\frac{2}{3}.$$

(A proof by induction is lurking in the wings.) Similarly,

$$\int_0^{\pi/2} \sin^{2n} x\,dx = \frac{2n-1}{2n}\cdot\ldots\cdot\frac{1}{2}\int_0^{\pi/2} \sin x\,dx$$

$$= \frac{2n-1}{2n}\cdot\ldots\cdot\frac{1}{2}\cdot\frac{\pi}{2}.$$

(c)

$$0 < \int_0^{\pi/2} \sin^{2n+1} x\,dx \le \int_0^{\pi/2} \sin^{2n} x\,dx \le \int_0^{\pi/2} \sin^{2n-1} x\,dx$$

so

$$1 \leq \frac{\int_0^{\pi/2} \sin^{2n} x\,dx}{\int_0^{\pi/2} \sin^{2n+1} dx} \leq \frac{\int_0^{\pi/2} \sin^{2n-1} x\,dx}{\int_0^{\pi/2} \sin^{2n+1} x\,dx}$$

$$= \frac{\int_0^{\pi/2} \sin^{2n-1} x\,dx}{\frac{2n}{2n+1}\int_0^{\pi/2} \sin^{2n-1} x\,dx}$$

$$= 1 + \frac{1}{2n}.$$

(d) If n is large, then

$$\frac{\sqrt{\pi}}{\sqrt{2}} \text{ is close to} \sqrt{\frac{2\cdot 2\cdot 4\cdot 4\cdot \ldots \cdot 2n\cdot 2n}{1\cdot 3\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)(2n+1)}} = \frac{2\cdot 4\cdot \ldots \cdot 2n}{\sqrt{2n+1}\cdot 1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}$$

$$= \frac{\sqrt{\frac{2n}{2n+1}}}{\sqrt{2}}\,\frac{1}{\sqrt{n}}\,\frac{2\cdot \ldots \cdot 2n}{1\cdot 3\cdot \ldots \cdot (2n-1)}.$$

Since $\sqrt{(2n)/(2n+1)}$ is close to 1 for large n, the result follows. [Wallis' procedure was quite different. He worked with the integral

$$\int_0^1 (1-x^2)^n\,dx$$

(which appears in Problem 27), hoping to recover, from the values obtained for natural numbers n, a formula for

$$\frac{\pi}{4} = \int_0^1 (1-x^2)^{1/2}\,dx.$$

Wallis first obtained the formula

$$\int_0^1 (1 - x^2)^n \, dx = \frac{2}{3}\cdot\frac{4}{5}\cdot\ldots\cdot\frac{2n}{2n + 1}$$

$$= \frac{(2\cdot4\cdot\ldots\cdot2n)^2}{2\cdot3\cdot4\cdot\ldots\cdot2n(2n + 1)} = \frac{2^n}{2n + 1}\,\frac{(n!)^2}{(2n)!},$$

(by what method I am not certain). He then reasoned that $\pi/4$ should be

$$\int_0^1 (1 - x^2)^{1/2} \, dx = \frac{2^1}{2}\,\frac{\left(\frac{1}{2}!\right)^2}{1!} = \left(\frac{1}{2}!\right)^2.$$

If we interpret $\frac{1}{2}!$ to mean $\Gamma(1/2)$, this agrees with Problem 29, but Wallis did not know of the gamma function (which was invented by Euler, guided principally by Wallis' work). Since $(2n)!/(n!)^2$ is the binomial coefficient $\binom{2n}{n}$, Wallis hoped to find $\frac{1}{2}!$ by finding $\binom{p + q}{p}$ for $p = q = 1/2$. Now

$$\binom{p + q}{p} = \frac{(p + q)(p + q - 1)\cdot\ldots\cdot(p + 1)}{q!},$$

and this makes sense even if p is not a natural number. Wallis therefore decided that

$$\binom{\frac{1}{2} + q}{\frac{1}{2}} = \frac{\left(\frac{1}{2} + q\right)\cdot\ldots\cdot\left(\frac{3}{2}\right)}{q!}.$$

With this interpretation of $\binom{p + q}{p}$ for $p = 1/2$, it is still true that

$$\binom{p + q + 1}{p} = \frac{p + q + 1}{q + 1}\binom{p + q}{p}$$

Denoting $\binom{\frac{1}{2} + q}{\frac{1}{2}}$ by W(q) this equation can be written

$$W(q + 1) = \frac{\frac{1}{2} + q + 1}{q + 1} W(q) = \frac{2q + 3}{2q + 2} W(q),$$

which leads to the table

| $q$ | $1$ | $2$ | $3$ | $\ldots$ |
|---|---|---|---|---|
| $W(q)$ | $\frac{3}{2}$ | $\frac{3}{2}\cdot\frac{5}{4}$ | $\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}$ | |

But, since W(1/2) should be $4/\pi$, Wallis also constructs the table

| $q$ | $\frac{1}{2}$ | $\frac{3}{2}$ | $\frac{5}{2}$ | $\ldots$ |
|---|---|---|---|---|
| $W(q)$ | $\frac{4}{\pi}$ | $\frac{4}{\pi}\cdot\frac{4}{3}$ | $\frac{4}{\pi}\cdot\frac{4}{3}\cdot\frac{6}{5}$ | |

Next Wallis notes that if $a_1$, $a_2$, $a_3$, $a_4$ are 4 successive values W(q), W(q + 1), W(q + 2), W(q + 3), appearing in either of these tables, then

$$\frac{a_2}{a_1} > \frac{a_3}{a_2} > \frac{a_4}{a_3} \qquad \left(\text{because } \frac{2q + 3}{2q + 2} > \frac{2q + 5}{2q + 4} > \frac{2q + 7}{2q + 6}\right),$$

which implies that

$$\sqrt{\frac{a_3}{a_1}} > \frac{a_3}{a_2} > \sqrt{\frac{a_4}{a_2}}.$$

Wallis then argues that this should still be true when $a_1$, $a_2$, $a_3$, $a_4$ are four successive values in a combined table where q is given both integer and half-integer values! Thus, taking as the four successive values $W(n + \frac{1}{2})$, $W(n)$, $W(n + \frac{3}{2})$, $W(n + 1)$, he obtains

$$\sqrt{\frac{\frac{4}{\pi}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdots\cdot\frac{2n+4}{2n+3}}{\frac{4}{\pi}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdots\cdot\frac{2n+2}{2n+1}}} > \frac{\frac{4}{\pi}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdots\cdot\frac{2n+2}{2n+1}}{\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}\cdots\cdot\frac{2n+1}{2n}}$$

$$> \sqrt{\frac{\frac{3}{2}\cdot\frac{5}{4}\cdots\cdot\frac{2n+3}{2n+2}}{\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}\cdots\cdot\frac{2n+1}{2n}}}$$

which yields simply

$$\sqrt{\frac{2n+4}{2n+3}} > \frac{4}{\pi}\cdot\left[\frac{2\cdot4\cdot4\cdot6\cdot6\cdot\ldots\cdot(2n)(2n)(2n+2)}{3\cdot3\cdot5\cdot5\cdot\ldots\cdot(2n+1)(2n+1)}\right] > \sqrt{\frac{2n+3}{2n+2}},$$

from which Wallis' product follows immediately.]

27. (a) Let $x = \cos u$, $dx = -\sin u \, du$. Then

$$\int_0^1 (1-x^2)^n \, dx = \int_{-\pi/2}^{0} (\sin^{2n} u)(-\sin u) \, du = \int_0^{\pi/2} \sin^{2n+1} u \, du$$

$$= \frac{2}{3}\cdot\frac{4}{5}\cdot\ldots\cdot\frac{2n}{2n+1} \qquad \text{by Problem 26.}$$

Now let $x = \cot u$, $dx = -\csc^2 u \, du$. Then

$$\int_0^{\infty} \frac{1}{(1+x^2)^n} \, dx = \int_{\pi/2}^{0} (\sin^{2n} u)\left(\frac{-1}{\sin^2 u}\right) du$$

$$= \int_0^{\pi/2} \sin^{2(n-1)} u \, du$$

$$= \frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\cdot\frac{2n-3}{2n-2} \qquad \text{by Problem 26.}$$

(b) If $f(y) = 1 - y$ and $g(y) = e^{-y}$, then $f(0) = g(0)$ and

$$f'(y) = -1 \leq -e^{-y} \quad \text{for } y \geq 0,$$

so $f(y) \leq g(y)$ for $y \geq 0$, i.e., $1 - y \leq e^{-y}$ for $y \geq 0$. So, in particular, $1 - x^2 \leq e^{-x^2}$ (for all $x$).

The second inequality follows from the inequality $1 + y \leq e^y$, which can be proved similarly (and has already appeared in Problem 17-21).

(c)

$$\int_0^1 (1 - x^2)^n \, dx \leq \int_0^1 e^{-nx^2} \, dx \leq \int_0^\infty e^{-nx^2} \, dx \leq \int_0^\infty \frac{1}{(1 + x^2)^n} \, dx,$$

so

$$\frac{2}{3} \cdot \dots \cdot \frac{2n}{2n + 1} \leq \int_0^1 e^{-nx^2} \, dx \leq \int_0^\infty e^{-nx^2} \, dx \leq \frac{\pi}{2} \cdot \frac{1}{2} \cdot \dots \cdot \frac{2n - 3}{2n - 2}.$$

Using the substitution $y = \sqrt{n}\, x$, $dx = 1/\sqrt{n}\, dy$, we obtain

$$\int_0^1 e^{-nx^2} \, dx = \frac{1}{\sqrt{n}} \int_0^{\sqrt{n}} e^{-y^2} \, dy,$$

$$\int_0^\infty e^{-nx^2} \, dx = \frac{1}{\sqrt{n}} \int_0^\infty e^{-y^2} \, dy,$$

from which the desired inequalities follow.

(d) It follows from Problem 26(d) that by choosing $n$ sufficiently large, the numbers

$$\frac{\pi}{2}\sqrt{n}\,\frac{1}{2}\cdot\frac{3}{4}\cdot\cdots\cdot\frac{2n-3}{2n-2}=\frac{\pi}{2}\sqrt{\frac{n}{n-1}}\left[\sqrt{n-1}\,\frac{1}{2}\cdot\frac{3}{4}\cdot\cdots\cdot\frac{2n-3}{2n-2}\right]$$

and

$$\sqrt{n}\,\frac{2}{3}\cdot\frac{4}{5}\cdot\cdots\cdot\frac{2n}{2n+1}=\frac{n}{2n+1}\left[\frac{1}{\sqrt{n}}\,\frac{2}{3}\cdot\cdots\cdot\frac{2n}{2n-1}\right]$$

can both be made as close as desired to

$$\frac{\pi}{2}\cdot 1\cdot\frac{1}{\sqrt{\pi}}=\frac{\sqrt{\pi}}{2}$$

and

$$\frac{1}{2}\cdot\sqrt{\pi}=\frac{\sqrt{\pi}}{2}.$$

28. (a)

$$\int_a^b \sin x\cdot\frac{1}{x}\,dx=-\cos x\cdot\frac{1}{x}\Big|_a^b-\int_a^b -\cos x\cdot-\frac{1}{x^2}\,dx$$

$$=\frac{\cos a}{a}-\frac{\cos b}{b}-\int_a^b\frac{\cos x}{x^2}\,dx.$$

In particular,

$$\int_1^\infty\frac{\sin x}{x}\,dx=\frac{\cos 1}{1}-\int_1^\infty\frac{\cos x}{x^2}\,dx;$$

the latter integral exists because the integral

$$\int_1^\infty\left|\frac{\cos x}{x^2}\right|dx\le\int_1^\infty\frac{1}{x^2}\,dx$$

exists (compare Theorem 22-4).

On the other hand, the integral

$$\int_0^1 \frac{\sin x}{x}\,dx$$

exists and equals $\int_0^1 f(x)\,dx$, where f is the continuous function with

$$f(x) = \begin{cases} 1, & x = 0 \\ \dfrac{\sin x}{x}, & x \neq 0. \end{cases}$$

(b) According to Problem 15-31,

$$\int_0^\pi \frac{\sin(n + 1/2)t}{\sin t/2}\,dt = \int_0^\pi (1 + 2\cos t + \dots + 2\cos nt)\,dt$$

$$= \pi.$$

(c) The hint is the whole answer, since the function

$$f(t) = \begin{cases} 0, & t = 0 \\ \dfrac{2}{t} - \dfrac{1}{\sin t/2}, & t \neq 0 \end{cases}$$

is integrable on $[0, \pi]$.

(d) From parts (b) and (c) we have

$$\lim_{\lambda \to \infty} \int_0^\pi \frac{2\sin(\lambda + 1/2)t}{t}\,dt = \lim_{\lambda \to \infty} \int_0^\pi \frac{\sin(\lambda + 1/2)t}{\sin t/2}\,dt = \pi.$$

Using the substitution $u = (\lambda + 1/2)t$, we have

$$\lim_{\lambda\to\infty}\int_0^{\pi}\frac{2\sin(\lambda+1/2)t}{t}\,dt = \lim_{\lambda\to\infty}\int_0^{(\lambda+1/2)\pi} 2\sin u\cdot\frac{(\lambda+1/2)}{u}\cdot\frac{du}{\lambda+1/2}$$

$$= 2\int_0^{\infty}\frac{\sin u}{u}\,du.$$

29. (a) Let $u = t^x$, $du = xt^{x-1}\,dt$. Then

$$\Gamma(x) = \int_0^{\infty} e^{-t}t^{x-1}\,dt = \int_0^{\infty} e^{-u^{1/x}}\,\frac{du}{x}$$

$$= \frac{1}{x}\int_0^{\infty} e^{-u^{1/x}}\,du.$$

(b)

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^{\infty} e^{-u^2}\,du$$

$$= \sqrt{\pi} \qquad \text{by Problem 27.}$$

# CHAPTER 19

1. (ii) $P_{3,0}(x) = 1 + x + x^2/2.$

(iv) $P_{2n,\pi}(x) = -1 + \frac{(x-\pi)^2}{2!} - \dots + \frac{(-1)^{n+1}(x-\pi)^{2n}}{(2n)!}.$

(vi) $P_{n,2}(x) = \log 2 + \frac{(x-2)}{2} - \frac{(x-2)^2}{2^2 2!} + \dots + \frac{(-1)^{n+1}(x-2)^n}{2^n n!}.$

(viii) $P_{4,1}(x) = 3 + 9(x-1) + \frac{26(x-1)^2}{2!} + \frac{66(x-1)^3}{3!} + \frac{120(x-1)^4}{4!}.$

(x) $P_{n,0}(x) = 1 - x + x^2 - x^3 + \dots + (-1)^n x^n.$

2. (ii) $160 + 50(x-3) - 10(x-3)^2 + (x-3)^4.$

(iv) $9a + 3b + c + (6a+b)(x-3) + a(x-3)^2.$

3. (ii)

$$\sum_{i=0}^{9} \frac{(-1)^i 2^{2i+1}}{(2i+1)!} \qquad \left(\text{since } \frac{2^{2n+2}}{(2n+2)!} \le 10^{-12} \quad \text{for } 2n+2 \ge 20, \text{ or } n \ge 9\right).$$

(iv)

$$\sum_{i=0}^{7} \frac{1}{i!} \qquad \left(\text{since } \frac{3}{(n+1)!} < 10^{-4} \quad \text{for } n+1 \ge 8, \text{ or } n \ge 7\right).$$

4. (i) To obtain

$$\frac{1}{(2n+2)!} < 10^{-(10^{10})} \qquad \text{or} \qquad (2n+2)! > 10^{10^{10}}$$

it certainly suffices to choose $2n + 2 = 10^{10^{10}}$; we can also choose $2n + 2 = 10^{10}$, since $(10^{10})!$ is clearly $> 10^{10^{10}}$. So one possible sum is

$$\sum_{i=0}^{\frac{10^{10}}{2}} \frac{(-1)^i}{(2i+1)!}.$$

(ii)

$$\sum_{i=0}^{1,000} \frac{1}{i!} \text{ (since surely } (1001)! > 3\cdot 10^{1,000}).$$

(iii) We need to find an n with

$$\frac{10^{2n+2}}{(2n+2)!} < 10^{-20}$$

Now

$$\frac{10^{100+k}}{(100+k)!} = \frac{10^{100}}{100!}\cdot\frac{10}{101}\cdot\frac{10}{102}\cdot\ldots\cdot\frac{10}{100+k} < \frac{10^{100}}{100!}\cdot\frac{1}{10^k},$$

so

$$\frac{10^{100+k}}{(100+k)!} < 10^{-20}$$

when

$$\frac{10^{100}}{100!}\cdot\frac{1}{10^{k}} < 10^{-20} \qquad \text{or} \qquad \frac{10^{120}}{100!} < 10^{k}.$$

This certainly happens for $k = 120$, so we can take $2n + 2 = 220$ or $n = 109$, giving the sum

$$\sum_{i=0}^{109} \frac{(-1)^{i}10^{2i+1}}{(2i+1)!}.$$

(iv)

$$\sum_{i=0}^{234} \frac{10^{i}}{i!} \qquad \left(\text{since } \frac{3^{10}10^{235}}{(235)!} < \frac{10^{5}\cdot 10^{100}}{(100)!}\cdot\frac{1}{10^{135}} < 10^{-30}\right).$$

(v)

$$\sum_{i=0}^{\frac{10^{10}}{2}} (-1)^{i} \frac{\left(\frac{1}{10}\right)^{2i+1}}{2i+1} \qquad \left(\text{since } \frac{\left(\frac{1}{10}\right)^{2n+3}}{2n+3} < 10^{-(10^{10})} \qquad \text{for } 2n + 3 = 10^{10}\right).$$

5. (a)

$$\arctan\frac{1}{2} + \arctan\frac{1}{3} = \arctan\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{6}}\right) = \arctan 1 = \frac{\pi}{4}.$$

Since

$$\arctan\frac{1}{5} + \arctan\frac{1}{5} = \arctan\left(\frac{\frac{1}{5}+\frac{1}{5}}{1-\frac{1}{25}}\right) = \arctan\frac{5}{12},$$

we have

$$4\arctan\frac{1}{5} = \arctan\left(\frac{\frac{5}{12}+\frac{5}{12}}{1-\frac{25}{144}}\right) = \frac{120}{119},$$

so

$$4 \arctan \frac{1}{5} - \arctan \frac{1}{239} = \arctan\left(\frac{\frac{120}{119} - \frac{1}{239}}{1 + \frac{120}{119}\cdot\frac{1}{239}}\right) = \arctan 1 = \frac{\pi}{4}.$$

(b) To compute $\pi$ with an error $< 10^{-6}$, we must compute $\pi/4$ with an error $< 10^{-6}/4$, so it suffices to compute arctan 1/5 and arctan 1/239 with an error $< 10^{-6}/20 = 10^{-7}/2$. Now

$$\arctan x = x - \frac{x^3}{3} + \dots + \frac{(-1)^n x^{2n+1}}{2n+1} + R, \qquad |R| \le \frac{|x|^{2n+3}}{2n+3}.$$

So for $x = 1/5$ and $x = 1/239$ we need

$$\frac{1}{(2n+3)5^{2n+3}} < \frac{1}{2\cdot 10^7},$$

$$\frac{1}{(2n+3)239^{2n+3}} < \frac{1}{2\cdot 10^7},$$

respectively. We can take $n = 4$ and $n = 0$, respectively. Sc $\pi$ is

$$16\left(\frac{1}{5} - \frac{1}{3\cdot 5^3} + \frac{1}{5\cdot 5^5} - \frac{1}{7\cdot 5^7} + \frac{1}{9\cdot 5^9}\right) - 4\left(\frac{1}{239}\right)$$

with an error $< 10^{-6}$. To find the first 5 decimals of $\pi$ we must convert each term in parentheses into a decimal. If we compute each one to 7 correct decimals, then we will introduce an extra error of at most $10^{-7}$. Since we actually have

$$\frac{1}{(2\cdot 4+3)5^{2\cdot 4+3}} < \frac{1}{3\cdot 10^7},$$

$$\frac{1}{3\cdot 239^3} < \frac{1}{3\cdot 10^7},$$

this extra error will be no problem. The calculations are as follows:

$$\begin{array}{rrcrr}
1/5 = & .20000000 & & & \\
1/5\cdot 5^5 = & .00006400 & & 1/3\cdot 5^3 = & .00266666 \\
1/9\cdot 5^9 = & .00000005 & & 1/7\cdot 5^7 = & .00000182 \\
\cline{2-2}\cline{5-5}
 & .20006405 & & & .00266848 \\
 & -.00266848 & \longleftarrow & & \\
\cline{2-2}
 & .19739557 & & & \\
 & \times 16 & & & \\
\cline{2-2}
 & 11843730 & & 1/239 = & .0041841 \\
 & 1973955 & & & \times 4 \\
\cline{2-2}\cline{5-5}
 & 3.1583280 & & & .0167364 \\
 & -\ .0167364 & \longleftarrow & & \\
\cline{2-2}\cline{2-2}
 & 3.1415916 & & &
\end{array}$$

This error in this result is $< 10^{-6}$; consequently we can be sure that 14159 are the first five decimals of $\pi$ (because of the fortunate circumstance that the next digit in our answer is not 9 !). The first ten decimals of $\pi$ are

$$3.1415926535.$$

6. Clearly

$$f^{(k)}(x) = \alpha(\alpha - 1)\cdot\ldots\cdot(\alpha - k + 1)(1 + x)^{\alpha-k},$$

so

$$P_{n,0}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k = \sum_{k=0}^{n} \frac{\alpha(\alpha - 1)\cdot\ldots\cdot(\alpha - k + 1)}{k!}x^k$$

$$= \sum_{k=0}^{n} \binom{\alpha}{k} x^k.$$

The Cauchy form of the remainder is

$$R_{n,0}(x) = \frac{\alpha(\alpha - 1)\cdot\ldots\cdot(\alpha - n)}{n!}(1 + t)^{\alpha-n-1}(x - t)(x - 0),$$

and the Lagrange form is

$$R_{n,0}(x) = \frac{\alpha(\alpha - 1)\cdot\ldots\cdot(\alpha - n)}{(n + 1)}(1 + t)^{\alpha-n-1}(x - 0)^{n+1}.$$

7. (ii) $c_i = \sum_{j=0}^{i} a_j b_{i-j}.$

(iv) $c_0 = 0$; $c_i = a_{i-1}/i$ for $i > 0$.

8. (a) Since

$$\lim_{x \to 0} \frac{R(x)}{x^{2n+1}} = 0,$$

we have

$$0 = \lim_{x \to 0} \frac{R(x^2)}{(x^2)^{2n+1}} = \lim_{x \to 0} \frac{R(x^2)}{x^{4n+2}}.$$

Now

$$\sin(x^2) = P(x^2) + R(x^2);$$

since $Q(x) = P(x^2)$ is a polynomial of degree $4n + 2$, it follows from the corollary to Theorem 3 that Q is the Taylor polynomial of degree $4n + 2$ for f at 0.

(b)

$$f^{(k)}(0) = \begin{cases} 0, & k \neq 4\ell + 2 \\ \dfrac{(-1)^{\ell}(4\ell + 2)!}{(2\ell + 1)!}, & k = 4\ell + 2. \end{cases}$$

(c)

$$f^{(k)}(0) = \begin{cases} 0, & k \neq n\ell \\ \dfrac{g^{(\ell)}(0)(n\ell)!}{\ell!}, & k = n\ell. \end{cases}$$

9.
$$\left|\int_0^x \frac{e^t}{n!}(x - t)^n\,dt\right| = \int_x^0 \frac{e^t}{n!}|x - t|^n\,dt$$
$$\leq \int_x^0 \frac{|x - t|^n}{n!}\,dt \qquad \text{since } e^x \leq 1 \quad \text{for } x \leq 0$$
$$= \frac{|x|^{n+1}}{(n + 1)!}.$$

10. For $-1 < x \leq t \leq 0$ we have
$$0 < 1 + x \leq 1 + t \leq 1,$$
$$0 \leq \frac{1}{1 + t} \leq \frac{1}{1 + x}.$$

So
$$\left|\int_0^x \frac{t^n}{1 + t}\,dt\right| \leq \int_x^0 \frac{|t|^n}{1 + x}\,dt \leq \frac{|x|^{n+1}}{(1 + x)(n + 1)}.$$

11. (a) By hypothesis,
$$-M(x - a)^n \leq g'(x) \leq M(x - a)^n \quad \text{for } x \geq a.$$

It follows from the Mean Value Theorem that
$$\frac{-M(x - a)^{n+1}}{n + 1} \leq g(x) - g(a) \leq \frac{M(x - a)^{n+1}}{n + 1},$$

i.e., that $|g(x) - g(a)| \leq M(x - a)^{n+1}/(n + 1)$. The case $x \leq a$ is treated similarly.

(b) For every $\epsilon > 0$ there is a $\delta > 0$ such that $|g'(x)/(x - a)^n| \leq \epsilon$ for $|x - a| < \delta$. This means that $|g'(x)| \leq \epsilon|x - a|^n$ for $|x - a| < \delta$. Part (a) implies that $|g(x) - g(a)| \leq \epsilon|x - a|^{n+1}/(n + 1)$ for $|x - a| < \delta$. Since this is true for every $\delta > 0$, it follows that

$$\lim_{x \to a} \frac{g(x) - g(a)}{(x - a)^{n+1}} = 0.$$

(c) Since

$$g(x) = f(x) - \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x - a)^i,$$

we have

$$\begin{aligned} g'(x) &= f'(x) - \sum_{i=1}^{n} \frac{f^{(i)}(a)}{(i - 1)!}(x - a)^{i-1} \\ &= f'(x) - \sum_{j=0}^{n-1} \frac{f^{(j+1)}(a)}{j!}(x - a)^j \\ &= f'(x) - \sum_{j=0}^{n-1} \frac{(f')^{(j)}(a)}{j!}(x - a)^j \\ &= f'(x) - P_{n-1,a,f'}(x). \end{aligned}$$

(d) Theorem 1 is true for $n = 1$, by the definition of $f'$. Now assume that Theorem 1 is true for $n - 1$, and all functions $f$ for which $f'(a),\dots,f^{(n-1)}(a)$ exist. If $g$ is a function for which $g'(a),\dots,g^{(n)}(a)$ exist, then $f = g'$ is a function for which $f'(a),\dots,f^{(n-1)}(a)$ exist. Consequently,

$$\lim_{x \to a} \frac{g'(x) - P_{n-1,a,g'}(x)}{(x-a)^{n-1}} = 0.$$

Since $(g - P_{n,a,g})' = g' - P_{n-1,a,g'}$, it follows from part (b) that

$$\lim_{x \to a} \frac{g(x) - P_{n,a,g}(x)}{(x-a)^{n}} = 0.$$

12. Suppose $|f^{(n+1)}|$ is bounded, by some M, on some interval around a. Then for x in this interval we have

$$|R_{n,a}(x)| = \frac{|f^{(n+1)}(t)|}{n!}|x-a|^{n+1},$$

so

$$\frac{|R_{n,a}(x)|}{|x-a|^n} \le M|x-a|,$$

so

$$\lim_{x \to a} \frac{R_{n,a}(x)}{(x-a)^n} = 0.$$

A similar proof works for the integral form of the remainder, and for the Cauchy form, if $|f^{(n+1)}|$ is assumed bounded.

13. Problem 13-27 implies that

$$R_{n,a}(x) = \int_a^x \frac{f^{(n+1)}(t)}{n!}(x - t)^n\,dt$$

$$= \frac{f^{(n+1)}(t)}{n!}(x - t)^n(x - a)$$

for some t in (a, x). Similarly, Problem 13-28, with $f^{(n+1)}/n!$ for f and $g(t) = (x - t)^n$, yields the Lagrange form. (In both cases, however, we begin with the extra assumption that $f^{(n+1)}$ is integrable.)

14. (a) This follows from

$$f(a + h) = f(a) + f'(a)h + \frac{f''(a)}{2}h^2 + R_{2,a}(h),$$

$$f(a - h) = f(a) - f'(a)h + \frac{f''(a)}{2}h^2 + R_{2,a}(-h),$$

since

$$\lim_{h\to 0} \frac{R_{2,a}(h)}{h^2} = \lim_{h\to 0} \frac{R_{2,a}(-h)}{h^2} = 0.$$

(b)

$$\lim_{h\to 0^+} \frac{f(0 + h) + f(0 - h) - 2f(0)}{h^2} = \lim_{h\to 0} \frac{h^2 - h^2 - 0}{h^2} = 0$$

and similarly for $\lim_{h\to 0^-}$.

15. If $f'' > 0$, then

$$f(x) = f(a) + f'(a)(x-a) + \int_a^x f''(t)(x-t)\,dt$$

$$> f(a) + f'(a)(x-a) \qquad \text{for } x \neq a,$$

which says that the graph of f at x lies above the tangent line through a.

16. The proof is almost exactly the same as the proof given in the text for the equation $f'' + f = 0$.

17. (a)

$$f^{(n+1)} = (f^{(n)})' = \sum_{j=0}^{n-1} a_j f^{(j+1)} = \left(\sum_{j=0}^{n-2} a_j f^{(j+1)}\right) + a_{n-1} f^{(n)}$$

$$= \sum_{j=0}^{n-1} a_{j-1} f^{(j)} + a_{n-1} \sum_{j=0}^{n-1} a_j f^{(j)}$$

$$= \sum_{j=0}^{n-1} (a_{j-1} + a_{n-1}a_j) f^{(j)}.$$

(b) Letting $a_{-1} = a_{-2} = 0$, we have

$$f^{(n+2)} = \sum_{j=0}^{n-1} (a_{j-1} + a_{n-1}a_j) f^{(j+1)}$$

$$= \sum_{j=0}^{n-2} (a_{j-1} + a_{n-1}a_j) f^{(j+1)} + (a_{n-2} + a_{n-1}{}^2) \sum_{j=0}^{n-1} a_j f^{(j)}$$

$$= \sum_{j=0}^{n-1} (a_{j-2} + a_{n-1}a_{j-1} + a_{n-2}a_j + a_{n-1}{}^2 a_j) f^{(j)}.$$

(c) From the equation

$$f^{(n+1)} = \sum_{j=0}^{n-1} b_j{}^1 f^{(j)}$$

we obtain

$$\begin{aligned} f^{(n+2)} &= \sum_{j=0}^{n-2} b_j{}^1 f^{(j+1)} + b_{n-1}{}^1 f^{(n)} \\ &= \sum_{j=0}^{n-1} b_{j-1}{}^1 f^{(j)} + b_{n-1}{}^1 \sum_{j=0}^{n-1} a_j f^{(j)} \\ &= \sum_{j=0}^{n-1} b_j{}^2 f^{(j)}, \end{aligned}$$

where

$$b_j{}^2 = b_{j-1}{}^1 + b_{n-1}{}^1 a_j,$$

$$|b_j{}^2| \leq |b_{j-1}{}^1| + |b_{n-1}{}^1||a_j| \leq 2N^2 + 2N^3 \leq 4N^3.$$

The general formula is proved similarly, by induction on k.

(d) Let $M = M_1 + \dots + M_{n-1}$, where

$$M_i = \sup\{|f^{(i)}(t)|: 0 \leq t \leq x\}.$$

(e) Clearly $f^{(k)}(0) = 0$ for all k. Then by Taylor's Theorem,

$$
\begin{aligned}
|f(x)| &= \left|\int_0^x \frac{f^{(n+k+1)}(t)}{(n+k)!}(x-t)^{n+k}\,dt\right| \\
&\leq \frac{M\cdot 2^{k+1}N^{k+2}|x|^{n+k+1}}{(n+k+1)!} \\
&\leq \frac{M\cdot|2Nx|^{n+k+1}}{(n+k+1)!}.
\end{aligned}
$$

Since $|2Nx|^{n+k+1}/(n+k+1)!$ can be made as small as desired by choosing k (and hence $n + k + 1$) sufficiently large, it follows that $f(x) = 0$.

(f) The difference $f = f_1 - f_2$ satisfies the same differential equation, and $f^{(j)}(0) = 0$ for $0 \leq j \leq n - 1$. So $f = 0$.

[In the case $n = 2$, the equations

$$
\begin{aligned}
f(0) &= c_1 + c_2 \\
f'(0) &= \alpha_1 c_1 + \alpha_2 c_2
\end{aligned}
$$

can always be solved if $\alpha_1 \neq \alpha_2$:

$$
c_1 = \frac{\alpha_2 f(0) - f'(0)}{\alpha_2 - \alpha_1}, \qquad c_2 = \frac{\alpha_1 f(0) - f'(0)}{\alpha_1 - \alpha_2}.
$$

The case $n = 3$ involves more complicated answers, but is equally straightforward. The general case, for those who know about determinants, depends upon the fact that the "Vandermonde determinant"

$$\det\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \vdots & \vdots & & \vdots \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \cdots & \alpha_n^{n-1} \end{pmatrix}$$

is non-zero if the $\alpha_i$ are distinct--in fact, it has the value $\prod_{i>j} \alpha_i - \alpha_j$.]

18. (a) Clearly f = 0 up to order 2 at 0. The second derivative $f''(0)$ does not exist because

$$f'(x) = \begin{cases} 0, & x = 0 \\ 4x^3 \sin 1/x^2 - x \cos 1/x^2, & x \neq 0, \end{cases}$$

and

$$\lim_{h \to 0} \frac{4h^3 \sin 1/h^2 - h \cos 1/h^2}{h}$$

does not exist.

(b) Apologies are in order. When this problem was first written, I had in mind a proof which upon closer examination turned out to be incorrect. Consequently, the hint given in part (b) is quite misleading. It leads to the equation

$$\frac{f'(a+h) - f'(a)}{h} = \frac{m(a)}{2} + \frac{m(a+h)}{2} + \frac{R_a(h)}{h^2} + \frac{R_{a+h}(a)}{h^2};$$

unfortunately, nothing is known about $\lim_{h \to 0} R_{a+h}(a)/h^2$. The correct proof turns out to be considerably more involved.

We begin by considering the special case where $m(a) = 0$ for all $a$. Writing equation (*) for $x = a + h$ and $x = a - h$, we then obtain

$$\lim_{h \to 0} \frac{f(a + h) + f(a - h) - 2f(a)}{h^2} = \lim_{h \to 0} \frac{R_a(h) + R_a(-h)}{h^2} = 0.$$

(Compare Problem 14). The limit on the left is called the Schwartz second derivative, and the following result, which solves the case $m = 0$, is known as Schwartz's Theorem.

If $f$ is continuous and

$$\lim_{h \to 0} \frac{f(x + h) + f(x - h) - 2f(x)}{h^2} = 0$$

for all $x$, then $f$ is linear.

Proof. Given two points $a < b$, define

$$\phi(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Then $\phi$ satisfies the same condition as $f$, and $\phi(a) = \phi(b)$. We will show that $\phi(x) = \phi(a)$ for all $x$ on $[a, b]$, thus proving that $f$ is linear on $[a, b]$. Since $a$ and $b$ are arbitrary, this will show that $f$ is linear.

Suppose that $\phi(x) > \phi(a)$ for some $x$ in $[a, b]$. Then for sufficiently small $\epsilon > 0$,

$$g(x) = \phi(x) - \epsilon(x - a)(b - x) > \phi(a).$$

Since the continuous function g satisfies $g(a) = \phi(a) = \phi(b) = g(b)$, it follows that g has a maximum point y in the open interval (a, b). Then

$$\frac{g(y + h) + g(y - h) - 2g(h)}{h^2} \leq 0.$$

But

$$\frac{g(y + h) + g(y - h) - 2g(y)}{h^2} = \frac{\phi(y + h) + \phi(y - h) - 2\phi(y)}{h^2} + 2\epsilon.$$

(The term $2\epsilon$ is the Schwartz second derivative of the function $\alpha(x) = -\epsilon(x - a)(b - x)$, which is the same as its ordinary second derivative.) This is a contradiction, since the right side approaches $2\epsilon$ as h approaches 0. Similarly, we cannot have $\phi(x) < \phi(a)$ for any x in [a, b].

The general case can now be deduced as follows. Since m is continuous, there is a function g with $g'' = m$. Then the function f - g satisfies (*) with m = 0. So f - g is a linear function. Consequently, $f'' = g'' = m$.

(c) If there is a continuous function m such that

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2 + \frac{m(a)}{3!}(x - a)^3 + R_a(x),$$

where $\lim_{x \to a} R_a(x)/(x - a)^3 = 0$, then $f'''(a) = m(a)$. The proof will follow closely along the lines of part (b). As in part (b), we can easily reduce the general case to the special case where $m = 0$.

We begin by noticing that if $f$ is a function which does have a third derivative, then

$$\lim_{h \to 0} \frac{f(x + h) - f(x - h) - 2hf'(x)}{h^3} = \frac{f'''(x)}{3};$$

the proof is similar to that for Problem 14, using the Taylor polynomial of degree 3 instead of the Taylor polynomial of degree 2. Of course, the same proof then shows that any function satisfying

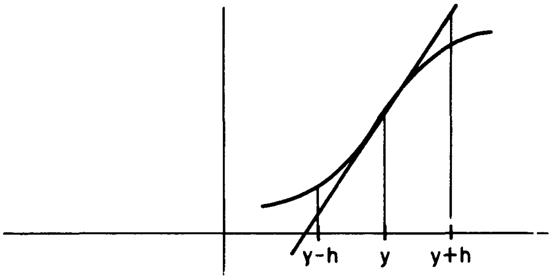
$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2 + R_a(x),$$

with $\lim_{x \to a} R_a(x)/(x - a)^3 = 0$, also satisfies

$$\lim_{h \to 0} \frac{f(x + h) - f(x - h) - 2hf'(x)}{h^3} = 0.$$

We will now show that any function with this property is a polynomial function of degree 2. It suffices to show that if $f'(a) = f'(b)$, then $f$ is constant on $[a, b]$, because, as in part (b), we can always subtract off an appropriate polynomial function of degree 2. To prove that $f$ is constant it suffices to prove that $f'(x) = 0$ for all $x$ in $[a, b]$. Suppose that this were not true and, to be specific, suppose that

$f'(x_0) > 0$ for some $x_0$ in $(a, b)$. Let $q$ be a polynomial function of degree 3 with $q'(a) = q'(b) = 0$ but $q'(x) > 0$ for all $x$ in $(a, b)$ (for example, $q(x) = -x^3/3 + (a + b)x^2/2 - abx$, so that $q'(x) = (x - a)(b - x)$.) Let $g(x) = f(x) - \epsilon q(x)$. Then $g'(x_0) = f'(x_0) - \epsilon q'(x_0)$ is $> 0$ for sufficiently small $\epsilon > 0$. Consequently, $g'$ will have a maximum at some point $y$ in $(a, b)$.



If $h > 0$, then by the Mean Value Theorem,

$$g(y + h) - g(y) \leq hg'(y),$$

$$g(y) - g(y - h) \leq hg'(y),$$

so

$$\frac{g(y + h) - g(y - h) - 2hg'(y)}{h^3} \leq 0.$$

The same inequality can also be derived for $h < 0$! But

$$\frac{g(y+h) - g(y-h) - 2hg'(y)}{h^3} = \frac{f(y+h) - f(y-h) - 2hf'(y)}{h^3} - \frac{q'''(y)\epsilon}{3}$$

$$= \frac{f(y+h) - f(y-h) - 2hf'(y)}{h^3} + \frac{2\epsilon}{3} .$$

This is a contradiction, since the right side approaches $2\epsilon/3$ as $h$ approaches 0.

1. (a) If $\alpha > 0$ is a solution of the equation

$$(1) \qquad a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 = 0,$$

then $\sqrt{\alpha}$ is a solution of the equation

$$a_n x^{2n} + a_{n-1} x^{2n-2} + \dots + a_0 = 0.$$

(b) If $\alpha$ satisfies (1), then $\alpha + r$ satisfies

$$a_n(x - r)^n + a_{n-1}(x - r)^{n-1} + \dots + a_0 = 0;$$

this equation, with rational coefficients, has the same solutions as the equation with integer coefficients that is obtained by multiplying through by a common denominator of the various coefficients.

Similarly, $\alpha r$ satisfies

$$a_n\left(\frac{x}{r}\right)^n + a_{n-1}\left(\frac{x}{r}\right)^{n-1} + \dots + a_0 = 0.$$

2. Since

$$(\sqrt{2} + \sqrt{3})^2 = 5 + 2\sqrt{6},$$

$$(\sqrt{2} + \sqrt{3})^4 = (5 + 2\sqrt{6})^2 = 49 + 20\sqrt{6},$$

it is clear that $\sqrt{2} + \sqrt{3}$ satisfies $x^4 - 10x^2 + 1 = 0$.

Since

$$[\sqrt{2}(1+\sqrt{3})]^2 = 2(4+2\sqrt{3}) = 8+4\sqrt{3},$$

$$[\sqrt{2}(1+\sqrt{3})]^4 = (8+4\sqrt{3})^2 = 112+64\sqrt{3},$$

clearly $\sqrt{2}(1+\sqrt{3})$ satisfies $x^4 - 16x^2 + 16 = 0$.

3. (a) If $f(p/q) = 0$, then

$$a_nx^n + a_{n-1}x^{n-1} + \dots + a_0 = (x - \frac{p}{q})(b_{n-1}x^{n-1} + \dots + b_0)$$

for some $b_0,\dots,b_{n-1}$, which will be rational numbers. Since $\alpha - p/q \neq 0$, we have

$$b_{n-1}\alpha^{n-1} + \dots + b_0 = 0,$$

contradicting the assumption about the minimal degree of the original polynomial.

(b) Clearly $f(p/q)$ can be written as a rational number of the form $r/q^n$. Since $f(p/q) \neq 0$, we have $|r| \geq 1$, so $|f(p/q)| \geq 1/q^n$.

(c) If $|\alpha - p/q| < 1$, then

$$f(p/q) = \frac{f(p/q) - f(\alpha)}{p/q - \alpha} = f'(x) \qquad \text{for } |x - \alpha| < 1$$
$$< M,$$

so

$$|\alpha - p/q| > |f(p/q)|/M > 1/(Mq^n).$$

4. If $\alpha$ satisfied a polynomial equation of degree n, then there would be some number c with $|\alpha - p/q| > c/q^n$ for all rational p/q. Now

$$\frac{1}{10^{1!}} + \frac{1}{10^{2!}} + \dots + \frac{1}{10^{k!}}$$

can be written as

$$\frac{p}{10^{k!}}$$

for some integer p, and

$$0 < \alpha - \frac{p}{10^{k!}} < \frac{2}{10^{(k+1)!}}.$$

Thus for every k we must have

$$\frac{c}{(10^{k!})^n} < \frac{2}{10^{(k+1)!}}$$

or

$$\frac{10^{(k+1)!}}{(10^{k!})^n} < \frac{2}{c},$$

or

$$\frac{(10^{k!})^{k+1}}{(10^{k!})^n} < \frac{2}{c},$$

or

$$(10^{k!})^{k+1-n} < \frac{2}{c},$$

which is clearly false for large enough n.

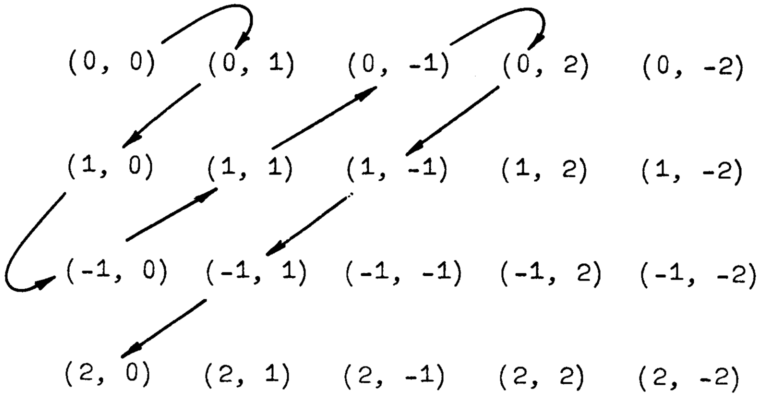
5. (a) If the elements of A and B can be arranged in the respective sequences $a_1, a_2, a_3, a_4, \ldots$ and $b_1, b_2, b_3, b_4, \ldots$, then the elements of $A \cup B$ can be arranged in the sequence

$$a_1, b_1, a_2, b_2, a_3, b_3, \ldots$$

(except that repetitions must be thrown away, if A and B have any elements in common).

(b) Arrange the positive rational numbers in a list by following the arrows (deleting repetitions).

(c) Follow the arrows in the picture



(d) Let the elements in $A_i$ be arranged in a list $a_1{}^i, a_2{}^i, a_3{}^i, \ldots$. Then the elements in $A_1 \cup A_2 \cup A_3 \cup \ldots$ can be arranged in the array

$$\begin{matrix} a_1^{\ 1} & a_2^{\ 1} & a_3^{\ 1} & a_4^{\ 1} \\ a_1^{\ 2} & a_2^{\ 2} & a_3^{\ 2} & a_4^{\ 2} \\ a_1^{\ 3} & a_2^{\ 3} & a_3^{\ 3} & a_4^{\ 3} \end{matrix}$$

Now use the same trick as in parts (b) and (c), deleting any repetitions.

(e) Apply part (d) with $A_i$ the set of all triples $(m, n, i)$. ($A_i$ is countable, by part (c).)

(f) If the set of all n-tuples is countable, then the set of all $(n + 1)$-tuples is seen to be countable by applying part (d) with $A_i$ the set of all $(n + 1)$-tuples $(a_1, a_2,\dots,a_n, i)$.

(g) (Polynomial functions with integer coefficients are meant.) Since every polynomial function $f(x) = a_n x^{n-1} + \dots + a_0$ of degree n can be described by an $(n + 1)$-tuple of integers $(a_0,\dots,a_n)$, these polynomial functions can be arranged in a list $p_1, p_2, p_3,\dots$ . For each $p_i$, let $\alpha_{i,1},\dots,\alpha_{i,n}$ be its roots (if there are fewer than n roots, choose 0 for the remaining $\alpha_{i,j}$). Then

$$\alpha_{1,1},\dots,\alpha_{1,n}, \alpha_{2,1},\dots,\alpha_{2,n}, \alpha_{3,1},\dots,\alpha_{3,n},\dots$$

is a list of all the desired roots. Now delete repetitions.

(h) Apply part (d) with $A_i$ the set of all roots of polynomial functions, with integer coefficients, of degree i.

6. If this number were in the list, it would be $\alpha_n$ for some n. But it cannot be $\alpha_n$, since it differs from $\alpha_n$ in the $n^{th}$ decimal place. (This tricky construction, and others modelled on it, is known as the Cantor diagonal method.)

7. (a) Suppose $0 < a_1 < \dots < a_n < 1$, and $\lim\limits_{x \to a_i^+} f(x) - \lim\limits_{x \to a_i^-} f(x) > \epsilon$. Choose

$$0 < a_1' < a_1 < a_1'' < a_2' < a_2 < a_2'' < \dots < a_n' < a_n < a_n'' < 1.$$

Then

$$f(a_i'') - f(a_i') > \epsilon,$$

so

$$f(1) - f(0) > \sum_{i=1}^{n} f(a_i'') - f(a_i') > n\epsilon,$$

so $n < [f(1) - f(0)]/\epsilon$.

(b) Let $A_n$ be the set of all a in [0, 1] with $\lim\limits_{x \to a^+} f(x) - \lim\limits_{x \to a^-} f(x) > 1/n$. Then $A_n$ is finite, so by Problem 5(d), $A_1 \cup A_2 \cup A_3 \cup \dots$ is countable.

8. (a) There are only countably many such intervals, since each interval is determined by a pair of rational numbers, and there are countably many rational numbers. Since each value f(x) can be described in terms of these intervals (as the maximum value on such an interval), there are only countably many values of f(x).

(b) If f happens to be continuous, then f cannot take on two values, for if it did, it would also take on all values inbetween, and hence an uncountable set of values.

(c) Like the statement, the proof is only a minor variant of part (a).

(d) This follows from part (c) in the same way that part (b) follows from part (a).

# CHAPTER 21

1. (ii)

$$\lim_{n\to\infty} \frac{n+3}{n^3+4} = \lim_{n\to\infty} \frac{\dfrac{1}{n^2} + \dfrac{3}{n^3}}{1 + \dfrac{4}{n^3}} = \frac{\lim\limits_{n\to\infty} \dfrac{1}{n^2} + \dfrac{3}{n^3}}{\lim\limits_{n\to\infty} 1 + \dfrac{4}{n^3}} = \frac{0}{1} = 0.$$

(iv) If n is even, then

$$\frac{n!}{n^n} = \frac{n(n-1)\cdot\ldots\cdot(n/2)!}{n^{n/2}\cdot n^{n/2}} \le \frac{(n/2)!}{n^{n/2}} \le \left(\frac{1}{2}\right)^{n/2};$$

similarly, if n is odd,. then

$$\frac{n!}{n^n} = \frac{n(n-1)\cdot\ldots\cdot[(n-1)/2]!}{n^{(n+1)/2}\cdot n^{(n-1)/2}} \le \frac{[(n-1)/2]!}{n^{(n-1)/2}} \le \left(\frac{1}{2}\right)^{(n-1)/2}.$$

(vi) $\lim\limits_{n\to\infty} (\log n)/n = 0$ (since $\lim\limits_{x\to\infty} (\log x)/x = 0$, by Problem 17-8(b)). So

$$\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} e^{(\log n)/n} = e^0 \text{ (by Theorem 1) } = 1.$$

(viii) Suppose $a \ge b$. Then $\sqrt[n]{a^n} \le \sqrt[n]{a^n + b^n} \le \sqrt[n]{a^n + a^n}$, i.e., $a \le \sqrt[n]{a^n + b^n} \le \sqrt[n]{2}\, a$, and $\lim\limits_{n\to\infty} \sqrt[n]{2} = 1$, by part (v). [It was necessary to assume $a, b \ge 0$ for this proof; in fact, if $a = 1$ and $b = -1$, then $\lim\limits_{n\to\infty} \sqrt[n]{a^n + b^n}$ does not exist.]

(x) According to Problem 2-6,

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} k^p}{n^{p+1}} = \lim_{n\to\infty} \frac{\frac{n^{p+1}}{p+1} + An^p + Bn^{p-1} + \ldots}{n^{p+1}}$$

$$= \lim_{n\to\infty} \frac{1}{p+1} + \frac{A}{n} + \frac{B}{n^2} + \ldots = \frac{1}{p+1}.$$

2. (a) The sequence $\{a_n\}$ must be eventually constant, that is, there is some $N$ such that all $a_n$ are equal for $n > N$.

(b) All convergent subsequences are of the form

$$a_1, \ldots, a_n, 1, 1, 1, 1, 1, \ldots$$

or

$$a_1, \ldots, a_n, -1, -1, -1, -1, \ldots,$$

where $a_1, \ldots, a_n$ is some finite sequence of 1's and -1's.

(c) All convergent subsequences are of the form

$$a_1, \ldots, a_n, m, m, m, m, \ldots,$$

where $a_1, \ldots, a_n$ and $m$ are natural numbers.

(d) All $\alpha$ in $[0, 1]$.

3. (a) Let $\{a_n\}$ be a Cauchy sequence, and suppose that $\lim_{j\to\infty} a_{n_j} = \ell$. For any $\epsilon > 0$, choose $J$ so that $|\ell - a_{n_j}| < \epsilon/2$ for $j > J$. Then choose $N$ so that $|a_n - a_m| < \epsilon/2$ for $n,m > N$. Let $N_0 = \max(N, n_J)$. If $n > N_0$,

then $|a_n - a_{n_{J+1}}| < \epsilon/2$ and $|a_{n_{J+1}} - \ell| < \epsilon/2$. Consequently, $|a_n - \ell| < \epsilon/2$.

(b) Suppose that $\lim_{n\to\infty} a_n = \ell$, and let $\{a_{n_j}\}$ be a subsequence of $\{a_n\}$. If $\epsilon > 0$, then there is some N such that $|\ell - a_n| < \epsilon$ for $n > N$. Since $n_1 < n_2 < n_3 < \dots$, there is some J such that $n_j > N$ for $j > J$. Thus $|\ell - a_{n_j}| < \epsilon$ for $j > J$. So $\lim_{j\to\infty} a_{n_j} = \ell$.

6. (ii) $\int_0^2 e^x\,dx = e^2 - 1$.

(iv) 0 (since

$$\frac{1}{n^2} + \frac{1}{(n+1)^2} + \dots + \frac{1}{(2n)^2} \le n\cdot\frac{1}{n^2} \le \frac{1}{n}).$$

(vi) $\int_0^1 \frac{1}{1+x^2}\,dx = \pi/4$.

7. (a) If $a = 1 + h$, then $a^n = (1+h)^n \ge 1 + nh$. Since $h > 0$, clearly $\lim_{n\to\infty} nh = \infty$.

(b) $\lim_{n\to\infty} a^n = 0$, because $\lim_{n\to\infty} 1/a^n = \infty$, by part (a).

(c) If $\sqrt[n]{a} = 1 + h$, then $a = (1+h)^n \ge 1 + nh$, so $h \le (a-1)/n$. Thus $1 \le \sqrt[n]{a} \le 1 + (a-1)/n$, so $\lim_{n\to\infty} \sqrt[n]{a} = 1$.

(d) $\lim_{n\to\infty} \sqrt[n]{a} = 1/(\lim_{n\to\infty} \sqrt[n]{1/a}) = 1$, by part (c).

(e) If $\sqrt[n]{n} = 1 + h$, then

$$n = (1 + h)^n \geq \frac{n(n - 1)}{2} h^2,$$

so

$$h \leq \sqrt{\frac{2}{n - 1}},$$

so

$$1 \leq \sqrt[n]{n} \leq 1 + \sqrt{\frac{2}{n - 1}},$$

so $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

8. (a) Suppose that $\lim_{n\to\infty} a_n = \ell$. Choose N so that $|a_n - \ell| < 1$ for $n > N$. Then $|a_n| < \max(|\ell| + 1, |a_1|,\dots,|a_N|)$ for all n.

(b) Choose N so that $|a_n - 0| < a_1$ for $n > N$. Then the maximum of $a_1, a_2,\dots,a_N$ is the maximum of $a_n$ for all n.

9. (a) This relation is equivalent to

$$\frac{1}{n + 1} < \int_n^{n+1} \frac{1}{x}\,dx < \frac{1}{n},$$

which is true because

$$\frac{1}{n + 1} < \frac{1}{x} < \frac{1}{n} \qquad \text{for } x \text{ in } (n, n + 1).$$

(b) Since

$$a_n - a_{n+1} = \log(n + 1) - \log n - \frac{1}{n + 1}$$

$$> 0 \qquad \text{by part (a),}$$

the sequence $\{a_n\}$ is decreasing.

To prove that $a_n \geq 0$, add the inequalities

$$\log(j + 1) - \log j < \frac{1}{j}$$

for $j = 1,\ldots,n - 1$, to obtain

$$\log n < 1 + \ldots + \frac{1}{n - 1}.$$

10. (a) Since f is increasing,

$$f(i) < \int_i^{i+1} f(x)\,dx < f(i + 1);$$

add these inequalities for $i = 1,\ldots,n - 1$.

(b) From part (a) we have

$$\log 1 + \ldots + \log(n - 1) < \int_1^n \log x\,dx < \log 2 + \ldots + \log n,$$

$$\log(n - 1)! < n \log n - n + 1 < \log n!,$$

$$(n - 1)! < \frac{n^n}{e^{n-1}} < n!.$$

So

$$\frac{n^n}{e^{n-1}} < n! < \frac{(n + 1)^{n+1}}{e^n} < \frac{n^{n+1}}{e^{n-1}}.$$

11. (a)

$$x^{b(x)} = x^{(\lim\limits_{n\to\infty} a_n(x))} = \lim_{n\to\infty} x^{a_n(x)}$$

$$= \lim_{n\to\infty} a_{n+1}(x) = b(x).$$

(b) The maximum value of $y^{1/y} = e^{(\log y)/y}$ occurs at the maximum value of $(\log y)/y$, which is e, by Problem 17-24.

(c) Clearly $a_1(x) = x \le e$. If $a_n(x) \le e$, then

$$a_{n+1}(x) = x^{a_n(x)} \le (e^{1/e})^e \le e.$$

(d) $b(\sqrt{2}) = 2$, since $\left(\sqrt{2}\right)^2 = 2$ (see part (a)).
$b(e^{1/e}) = e$.

(e) Part (a) shows that if $x = y^{1/y}$, then $b(x) = y$. The condition $x = y^{1/y}$ is equivalent to $\log x = (\log y)/y$. Using the function $f_1$ defined in Problem 17-24(f), this can be written

$$b(x) = f_1^{-1}(\log x).$$

So b is differentiable on $(0, e^{1/e})$, and

$$b'(x) = \frac{1}{f_1'(f_1^{-1}(\log x))} \cdot \frac{1}{x} = \frac{1}{xf_1'(b(x))}$$

$$= \frac{[b(x)]^2}{x(1 - \log b(x))}.$$

(f) Differentiating $x^{b(x)} = b(x)$ yields

$$x^{b(x)}\left[\frac{b(x)}{x} + b'(x)\log x\right] = b'(x),$$

$$b(x)\left[\frac{b(x)}{x} + b'(x)\log x\right] = b'(x),$$

$$b'(x) = \frac{[b(x)]^2}{x(1 - b(x)\log x)}.$$

Since $x^{b(x)} = b(x)$, we have $b(x)\log x = \log b(x)$, so

$$b'(x) = \frac{[b(x)]^2}{x(1 - \log b(x))}.$$

12. If $\epsilon > 0$, pick $N$ so that $|a_n - \ell| < \epsilon$ for $n \geq N$. Then

$$|a_N + a_{N+1} + \dots + a_{N+M} - M\ell| < \epsilon M,$$

so

$$\left|\frac{1}{N+M}[a_N + a_{N+1} + \dots + a_{N+M}] - \frac{M\ell}{N+M}\right| < \frac{\epsilon M}{N+M} < \epsilon.$$

Choose $M$ so that

$$\left|\frac{M\ell}{N+M} - \ell\right| < \epsilon \qquad \text{and} \qquad \left|\frac{1}{N+M}[a_1 + \dots + a_N]\right| < \epsilon.$$

Then

$$\left|\frac{1}{N+M}[a_1 + \dots + a_{N+M}] - \ell\right| < 3\epsilon.$$

13. If $\epsilon > 0$, pick $N$ so that $|a_{n+1}/a_n - \ell| < \epsilon$ for $n \geq N$. Then

$$\ell - \epsilon < \frac{a_{n+1}}{a_n} < \ell + \epsilon \qquad \text{for } n \geq N,$$

so

$$(\ell - \epsilon)^m < \frac{a_{n+m}}{a_{n+m-1}} \cdot \frac{a_{n+m-1}}{a_{n+m-2}} \cdots \frac{a_{n+1}}{a_n} < (\ell + \epsilon)^m,$$

so

$$\left| \sqrt[m]{\frac{a_{n+m}}{a_n}} - \ell \right| < \epsilon.$$

Now

$$\sqrt[n+m]{a_{n+m}} = \sqrt[n+m]{\frac{a_{n+m}}{a_n}} \cdot \sqrt[n+m]{a_n}$$

$$= \left( \sqrt[m]{\frac{a_{n+m}}{a_n}} \right)^{m/n+m} \cdot \sqrt[n+m]{a_n}.$$

Since $\lim\limits_{m \to \infty} \sqrt[n+m]{a_n} = 1$, it follows that $\sqrt[n+m]{a_{n+m}}$ can be made within $2\epsilon$ by choosing m sufficiently large.

14. (a) Suppose $\lim\limits_{n \to \infty} a_n = \ell > 1$. Since $\ell - 1 > 0$, there would be some n with $|\ell - a_n| < \ell - 1$, and hence $a_n > 1$, a contradiction. Similarly, we cannot have $\ell < 0$.

(b) $a_n = 1/n$.

15. Let us denote

$$\underbrace{f(f(f\ldots f}_{k \text{ times}}(x)\ldots))$$

by $f^k(x)$. Then by Theorem 1,

$$f(\ell) = f(\lim_{k\to\infty} f^k(x)) = \lim_{k\to\infty} f(f^{(k)}(x))$$
$$= \lim_{k\to\infty} f^{(k+1)}(x) = \ell.$$

16. (a) Suppose $f(x) > x$. Since $f$ is increasing, $f(f(x)) > f(x)$. Consequently, $f(f(f(x))) > f(f(x))$, etc. Thus the sequence $x, f(x), f(f(x)),\ldots$ is increasing, and bounded by 1, so it has a limit. The proof when $x < f(x)$ is similar.

(b) There is some $m$ with $g(m) = m$ (by Problem 7-11). According to part (a), the sequence $f^k(m)$ has a limit $\ell$, which is a fixed point for $f$ (using the notation introduced in the solution to Problem 15). Moreover,

$$f^k(m) = f^k(g(m)) = g(f^k(m)),$$

since $f\circ g = g\circ f$. Hence, by Theorem 1,

$$\ell = \lim_{k\to\infty} f^k(m) = \lim_{k\to\infty} g(f^k(m)) = g(\lim_{k\to\infty} f^k(m)) = g(\ell).$$

(c) At the time I wrote part (c), the answer to this problem was still unknown. A few weeks later there appeared, in the Notices of the

American Mathematical Society, Volume 14, Number 2, two independent announcements that the answer is "no". Which all goes to show, I suppose, that the problem was pretty silly in the first place.

17. (a)

$$\begin{aligned} c^m + c^{m+1} + \dots + c^n &= c^m(1 + c + \dots + c^{n-m}) \\ &= \frac{c^m(1 - c^{n-m+1})}{1 - c} \\ &= \frac{c^m - c^{n+1}}{1 - c}. \end{aligned}$$

(b) Since $|c| < 1$, we have $\lim\limits_{m\to\infty} c^m = \lim\limits_{n\to\infty} c^{n+1} = 0$.

(c)

$$\begin{aligned} |x_n - x_m| &= |(x_n - x_{n+1}) + (x_{n+1} - x_{n+2}) + \dots + (x_{m-1} - x_m)| \\ &\le |x_n - x_{n+1}| + |x_{n+1} - x_{n-2}| + \dots + |x_{m-1} - x_m| \\ &\le c^n + \dots + c^{m-1}, \end{aligned}$$

so $\lim\limits_{m,n\to\infty} |x_n - x_m| = 0$, by part (b).

18. (a) If $c = 0$, then $f$ is constant, so continuous. If $c \ne 0$ and $\epsilon > 0$, then $|f(x) - f(a)| < \epsilon$ for $|x - a| < \epsilon/c$.

(b) If $f(\ell) = \ell$ and $f(m) = m$, then

$$|\ell - m| = |f(\ell) - f(m)| \le c|\ell - m|,$$

so $\ell = m$, since $c < 1$.

(c) If

$$x_n = f^n(x) = \underbrace{f(f(f\ldots f}_{n \text{ times}}(x)\ldots)),$$

then

$$|x_n - x_{n+1}| = |f(x_{n-1}) - f(x_n)| \le c|x_{n-1} - x_n| \le c^2|x_{n-1} - x_{n-2}|$$
$$\le \ldots \le c^{n-1}|x_2 - x_1|.$$

Consequently, Problem 17(c) implies that $\{x_n\}$ is a Cauchy sequence, so converges. It converges to a fixed point, by Problem 15.

19. (a) Clearly $\{y_n\}$ is decreasing, and bounded below (by a lower bound for $\{x_n\}$).

(b) (i) 0.

(ii) 0.

(iii) 1.

(iv) 1.

(c) $\underline{\lim}_{n\to\infty} x_n = \lim_{n\to\infty} z_n$, where $z_n = \inf\{x_n, x_{n+1}, x_{n+2}, \ldots\}$. Since $z_n \le y_n$ for each n, it is clear that $\underline{\lim}_{n\to\infty} x_n \le \overline{\lim}_{n\to\infty} x_n$.

(d) Suppose first that $\lim\limits_{n\to\infty} x_n = \ell$. If $\epsilon > 0$ there is some N with $|x_n - \ell| < \epsilon$ for $n \geq N$. So $x_N < \ell + \epsilon$, $x_{N+1} < \ell + \epsilon, \ldots$, so $y_N \leq \ell + \epsilon$. Similarly $z_N \geq \ell - \epsilon$. Since this is true for all $\epsilon > 0$, we have $\underline{\lim}_{n\to\infty} x_n = \ell = \overline{\lim}_{n\to\infty} x_n$.

Conversely, suppose that $\underline{\lim}_{n\to\infty} x_n = \overline{\lim}_{n\to\infty} x_n = \ell$. Then for any $\epsilon > 0$ there is some N with $\ell - \epsilon < z_N < y_N < \ell + \epsilon$. This implies that $\ell - \epsilon < x_n < \ell + \epsilon$ for every $n \geq N$.

(e) Let $\ell = \overline{\lim}_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n$. If $a < \ell$, then $a < y_n$ for all n: Consequently, $a < x_n$ for infinitely many $x_n$, so a is not an almost upper bound for A. On the other hand, if $a > \ell$, then $a > y_n$ for all but finitely many n. Consequently, $a > x_n$ for all but finitely many n, so a is an almost upper bound. Thus, $\ell$ is the greatest lower bound of all almost upper bounds of A.

20. (a) Let $s_k = a_1 + \ldots + a_k$. Then

$$\begin{aligned} a_1b_1 + \ldots + a_nb_n &= b_1s_1 + b_2(s_2 - s_1) + \ldots + b_n(s_n - s_{n-1}) \\ &= s_1(b_1 - b_2) + s_2(b_2 - b_3) + \ldots + s_{n-1}(b_{n-1} - b_n) + s_nb_n. \end{aligned}$$

Since each $b_i - b_{i-1} \geq 0$, and $m \leq s_i \leq M$ for each i, we obtain

$$\begin{aligned} &m(b_1 - b_2) + m(b_2 - b_3) + \ldots + m(b_{n-1} - b_n) + mb_n \\ &\qquad \leq a_1b_1 + \ldots + a_nb_n \\ &\qquad\qquad \leq M(b_1 - b_2) + M(b_2 - b_3) + \ldots + M(b_{n-1} - b_n) + Mb_n, \end{aligned}$$

or

$$mb_1 \leq a_1b_1 + \dots + a_nb_n \leq Mb_1.$$

(b) Apply part (a) to the series $a_k, a_{k+1}, a_{k+2}, \dots$ and $b_k, b_{k+1}, b_{k+2}, \dots$ .

21. (a) (i) 0.

(ii) 0 and $1/n$ for each natural number $n$.

(iii) $-1$ and $1$.

(iv) No limit points.

(v) All real numbers.

(b) If there are infinitely many points $a$ of $A$ satisfying $|x - a| < \epsilon$, then there is surely one such $a$ with $a \neq x$. Conversely, if there were only finitely many such points $a_1, \dots, a_n$ and $\epsilon_1 > 0$ is the minimum of all those $|x - a_i|$ which are $\neq 0$, then there would be no points $a$ in $A$ satisfying $|x - a| < \epsilon_1$.

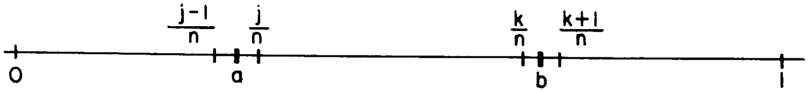
(c) For any $\epsilon > 0$, the number $\overline{\lim}\, A - \epsilon$ is not an almost upper bound of $A$, so there are infinitely many numbers $y$ in $A$ with $y > \overline{\lim}\, A - \epsilon$. Moreover, there cannot be infinitely many such numbers $y$ with $y > \overline{\lim}\, A + \epsilon$, for in this case no numbers between $\overline{\lim}\, A$ and $\overline{\lim}\, A + \epsilon$ could be almost upper bounds of $A$, so $\overline{\lim}\, A + \epsilon$ would be a larger lower bound for the set of all almost upper bounds. This shows that there are infinitely many numbers y in A between $\overline{\lim}\, A - \epsilon$ and $\overline{\lim}\, A + \epsilon$. Consequently, $\overline{\lim}\, A$ is a limit point of A. If there were another limit point $\alpha > \overline{\lim}\, A$, then no number less than $\alpha$ could be an almost upper bound, so $\alpha$ would be a lower bound for the set of almost upper bounds, a contradiction. The proof for $\underline{\lim}\, A$ is similar.

(d) Choose distinct points $x_1, x_2, x_3, \ldots$ in A. The sequence $\{x_n\}$ is bounded, since A is contained in $[a, b]$. So there is a convergent subsequence $\{x_{n_j}\}$. Let $\ell = \lim_{j \to \infty} x_{n_j}$. For any $\epsilon > 0$ there is some J such that $|\ell - x_{n_j}| < \epsilon$ for all $j > J$. Since the $x_{n_j}$ are distinct, this shows that there are infinitely many a in A with $|\ell - a| < \epsilon$.

(e) Choose a sequence of intervals $I_1, I_2, I_3, \ldots$ with $I_1 = [a, b]$, and each $I_{j+1}$ a half of $I_j$, such that each $I_j$ contains infinitely many points of A. If x is the point in all $I_j$, then x is a limit point of A.

22. Choose $x_n$ with $f(x_n) > n$. There is a subsequence $x_{n_j}$ which converges to a point x, which is in $[a, b]$. Thus for every $\epsilon > 0$ there are infinitely many $x_{n_j}$ with $|x - x_{n_j}| < \epsilon$, and consequently f is unbounded on $[x - \epsilon, x + \epsilon]$, contradicting the fact that f is continuous at x.

23. (a) Let $\#(n)$ be the number of j for which $j/n$ is in $[a, b]$. To estimate $\#(n)$, let $j/n$ be the smallest such fraction in $[a, b]$ and $k/n$ the largest. Then $(j - 1)/n < a \le j/n$ and $k/n \le b < (k + 1)/n$.



So

$$\frac{k}{n} - \frac{j}{n} \leq b - a < \frac{k+1}{n} - \frac{j-1}{n},$$

$$k - j \leq n(b - a) < k - j + 2.$$

Since $\#(n) = k - j + 1$, we also have

$$k - j < \#(n) < k - j + 2,$$

so

$$|\#(n) - n(b - a)| < 2.$$

Adding these inequalities for $1,\dots,n$, we obtain

$$|\#(1) + \dots + \#(n) - [1 + \dots + n](b - a)| < 2n.$$

Consequently,

$$\left|\frac{\#(1) + \dots + \#(n)}{1 + \dots + n} - (b - a)\right| < \frac{2n}{1 + \dots + n}.$$

Since $\lim\limits_{n\to\infty} 2n/(1 + \dots + n) = 0$, this shows that

$$\frac{\#(1) + \dots + \#(n)}{1 + \dots + n} \text{ approaches } (b - a).$$

Of course, $\#(1) + \dots + \#(n) = N(1 + \dots + n; a, b)$. For an arbitrary number $m$, let $n$ be the largest number with $1 + \dots + n \leq m$. Then

$$m - (1 + \ldots + n) \leq n.$$

Clearly,

$$|N(m; a, b) - [\#(1) + \ldots + \#(n)]| \leq m - (1 + \ldots + n) \leq n.$$

Consequently

$$(1) \quad \left|\frac{N(m; a, b)}{m} - \frac{[\#(1) + \ldots + \#(n)]}{m}\right| \leq \frac{n}{m} \leq \frac{n}{1 + \ldots + n} \to 0, \text{ as } n \to \infty.$$

Moreover,

$$\frac{\#(1) + \ldots + \#(n)}{m} = \frac{\#(1) + \ldots + \#(n)}{1 + \ldots + n} \cdot \frac{1 + \ldots + n}{m};$$

since

$$\frac{1 + \ldots + n}{1 + \ldots + (n + 1)} \leq \frac{1 + \ldots + n}{m} \leq \frac{1 + \ldots + n}{1 + \ldots + n} = 1,$$

it follows that $[\#(1) + \ldots + \#(n)]/m$ can be made as close to $[\#(1) + \ldots + \#(n)]/(1 + \ldots + n)$ as desired by choosing m (and hence n) sufficiently large. Since the latter expression can be made as close as desired to b - a by choosing m sufficiently large, it follows from (1) that $\lim_{m \to \infty} N(m; a, b)/m = b - a$.

(b) Consider the special case where $s(x) = c$ for x in [a, b] and $s(x) = 0$ for other x in [0, 1]. Then

$$\lim_{n \to \infty} \frac{s(a_1) + \ldots + s(a_n)}{n} = \lim_{n \to \infty} c \cdot \frac{N(n; a, b)}{n} = c(b - a) = \int_0^1 s.$$

This holds, in particular, when $a = b$. A similar proof works when $s(x) = c$ for $x$ in $(a, b)$. Any step function $s$ can be written $s = s_1 + \dots + s_m$ where $s_i$ is one of these special kinds. Then

$$\int_0^1 s = \sum_{i=1}^{m} \int_0^1 s_i = \sum_{i=1}^{m} \lim_{n\to\infty} \frac{s_i(a_1) + \dots + s_i(a_n)}{n} = \lim_{n\to\infty} \frac{s(a_1) + \dots + s(a_n)}{n}.$$

(c) Let $\epsilon > 0$. According to Problem 13-17, there is a step function $s \le f$ with $\int_a^b [f - s] < \epsilon$. Thus for sufficiently large $n$,

$$-\epsilon + \int_a^b f < \int_a^b s < \frac{s(a_1) + \dots + s(a_n)}{n} + \epsilon \le \frac{f(a_1) + \dots + f(a_n)}{n} + \epsilon.$$

Similarly, since there is a step function $s \ge f$ with $\int_a^b [s - f] < \epsilon$, we have

$$-2\epsilon + \int_a^b f < \frac{f(a_1) + \dots + f(a_n)}{n} < 2\epsilon + \int_a^b f.$$

for sufficiently large $n$.

24. (a) If there were infinitely many such points $a$ in $[0, 1]$, then the set of all such points would have a limit point $x$ in $[0, 1]$. For every $\delta > 0$ there would be some $a$ with $|a - x| < \delta/2$ and $|\lim_{y\to a} f(y) - f(a)| > \epsilon$. Consequently there would be $a'$ with $|a' - a| < \delta/2$ (and consequently $|a' - x| < \delta$) such that $|f(a') - f(a)| > \epsilon$. But since $\lim_{y\to x} f(x) = \ell$ for some $\ell$, there is some $\delta > 0$ such that $|f(y) - \ell| < \epsilon/2$ for

$|y - x| < \delta$. In particular, if $|a - x| < \delta$ and $|a' - x| < \delta$, then $|f(a) - f(a')| \leq |f(a) - \ell| + |f(a') - \ell| < \epsilon$, a contradiction.

(b) By part (a), the set $A_n$ of points a where $|\lim_{y \to a} f(y) - f(a)| > 1/n$ is finite. By Problem 20-5, the union $A_1 \cup A_2 \cup A_3 \cup \ldots$ is countable. This union is the same as the set of all points a at which f is discontinuous.

# CHAPTER 22

1. (ii) Convergent, by Leibnitz's Theorem. The series is not absolutely convergent, since

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots > \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \dots = \frac{1}{2}(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots).$$

(iv) Convergent, by Leibnitz's Theorem. (The function $f(x) = (\log x)/x$ is decreasing for $x \geq e$, since $f'(x) = (1 - \log x)/x^2$.) The series is not absolutely convergent (see (viii)).

(vi) Convergent, since

$$\frac{1}{\sqrt[3]{n^2 + 1}} \leq \frac{1}{n^{2/3}}.$$

(viii) Divergent, since

$$\int_1^N \frac{\log x}{x}\, dx = \frac{(\log N)^2}{2} \to \infty \text{ as } N \to \infty,$$

and $f(x) = (\log x)/x$ is decreasing for $x \geq e$ (see (iv)).

(x) Divergent, since

$$\frac{1}{(\log n)^k} > \frac{1}{n}$$

for sufficiently large $n$ (Problem 17-8).

(xii) (Absolutely) convergent, by (xi).

(xiv) Divergent, since

$$\sin\frac{1}{n} > \frac{1}{2n},$$

for sufficiently large n.

(xvi) Convergent, since

$$\int_2^N \frac{1}{x(\log x)^2}\,dx = -\frac{1}{\log N} + \frac{1}{\log 2} \to \frac{1}{\log 2} \text{ as } N \to \infty.$$

(xviii) Convergent, since

$$\lim_{n\to\infty}\frac{(n+1)!/(n+1)^{n+1}}{n!/n^n} = \lim_{n\to\infty}\frac{(n+1)n^n}{(n+1)^{n+1}} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e},$$

by Problem 17-12.

(xx) Divergent, since

$$\lim_{n\to\infty}\frac{3^{n+1}(n+1)!/(n+1)^{n+1}}{3^n n!/n^n} = \lim_{n\to\infty}\frac{3(n+1)n^n}{(n+1)^{n+1}} = \lim_{n\to\infty}\frac{3}{\left(1+\frac{1}{n}\right)^n} = \frac{3}{e},$$

by Problem 17-12.

2. (a) According to Problem 21-10,

$$\frac{e^n n!}{n^n} > e,$$

so the series certainly diverges.

(b) Since

$$\lim_{n\to\infty} \frac{(n+1)^{n+1}/a^{n+1}(n+1)!}{n^n/a^n n!} = \lim_{n\to\infty} \frac{a(n+1)^{n+1}}{(n+1)n^n} = \lim_{n\to\infty} a\left(1+\frac{1}{n}\right)^n = ae,$$

the series converges for $a < 1/e$ and diverges for $a > 1/e$. By Problem 21-10,

$$\frac{n^n}{e^n n!} = \frac{n^n}{(n+1)^{n+1}} \cdot \frac{(n+1)^{n+1}}{e^n n!} > \frac{n^n}{(n+1)^{n+1}}$$

$$= \frac{1}{n+1}\left(\frac{n}{n+1}\right)^n = \frac{1}{(n+1)\left(1+\frac{1}{n}\right)^n} > \frac{2}{e(n+1)}$$

for sufficiently large n, so $\sum_{n=1}^{\infty} n^n/e^n n!$ diverges.

3. (a) The function $f(y) = e^y/y^y$ is decreasing for $y \geq 1$, since

$$f'(y) = \frac{y^y e^y - e^y y^y(1+\log y)}{y^{2y}} = \frac{e^y}{y^y}(-\log y).$$

Now the series $\sum_{n=1}^{\infty} (e/n)^n$ clearly converges, since $(e/n)^n \leq e^2/n^2$ for $n \geq 2$, so the integral also converges.

(b) Since $f(x) = (\log x)^{-\log x}$ is clearly decreasing for $x \geq 1$, the series converges if $\int_0^{\infty} (\log x)^{-\log x}\,dx$ exists. The substitution $y = \log x$, $dx = e^y\,dy$, changes this integral to

$$\int_1^\infty \frac{e^y}{y^y}\,dy,$$

which exists, by part (a).

(c) The substitution $y = \log x$, $dx = e^y\,dy$ changes $\int_1^\infty (\log x)^{-\log(\log x)}\,dx$ to

$$\int_1^\infty \frac{e^y}{y^{\log y}}\,dy.$$

Now

$$\frac{e^y}{y^{\log y}} = \frac{e^y}{e^{(\log y)^2}} = e^{y-(\log y)^2} = e^{y(1-(\log y)^2/y)}.$$

Since $\lim\limits_{y\to\infty} (\log y)^2/y = 0$ (Problem 17-8), it follows that $e^y/y^{\log y}$ is close to $e^y$ for large $y$, so the integral certainly diverges.

4. (b) Define $\{a_n\}$ inductively as follows:

$$a_1 = [10x],$$

$$a_n = [10^n x - (10^{n-1}a_1 + \dots + 10a_{n-1})].$$

For each $n$ we have

$$0 \le 10^n x - (10^{n-1}a_1 + \dots + 10a_{n-1}) - a_n < 1,$$

so

$$(*) \qquad 0 \le 10^{n+1}x - (10^n a_1 + \dots + 10^2 a_{n-1} + 10a_n) < 10,$$

so $0 \le a_{n+1} \le 9$ for each n. Moreover, from (*) we have

$$0 \le x - (a_1 10^{-1} + a_2 10^{-2} + \dots + 10^{-n} a_n) < 10^{-n},$$

so $x = \sum_{n=1}^{\infty} a_n 10^{-n}$.

(c) Let $\alpha = 10^k a_1 + \dots + 10a_{k-1} + a_k$. Then

$$\begin{aligned}\sum_{n=1}^{\infty} a_n 10^{-n} &= \frac{\alpha}{10^k} + \frac{\alpha}{10^{2k}} + \frac{\alpha}{10^{3k}} + \dots \\ &= \frac{\alpha}{10^k}\left[1 + \frac{1}{10} + \left(\frac{1}{10}\right)^2 + \dots\right] \\ &= \frac{\alpha}{10^k} \cdot \frac{1}{1 - \frac{1}{10}} \\ &= \frac{9\alpha}{10^{k-1}}.\end{aligned}$$

(d) The number $a_n$ in part (b) satisfies

$$0 \le \frac{10^n p}{q} - (10^{n-1} a_1 + \dots + 10a_{n-1} + a_n) < 1.$$

Now $10^n p/q$ can be written as $k + r/q$ where k is an integer and $0 \le r < q - 1$. In this case $a_{n+1} = [10r/q]$. Since there are at most q different fractions $r/q$, there will have to be some m and n with $m > n$ and $a_{n+1} = [10r/q] = a_{m+1}$. It is easy to see that we will then have $a_{n+2} = a_{m+2}$, etc.

5. The proof of Leibnitz's Theorem shows that if N is even, then

$$s_N \leq \sum_{n=1}^{\infty} (-1)^{n+1} a_n \leq s_{N+1},$$

so $\left| \sum_{n=1}^{\infty} (-1)^{n+1} a_n - s_N \right| \leq s_{N+1} - s_N = a_{N+1} \leq a_N$. (Strict inequality holds unless $s_N = s_{N+1}$, or $a_{N+1} = 0$.) The proof is similar if N is odd.

6. There is some N such that $c/2 < a_n/b_n < 2c$ for $n > N$. Hence

$$b_n < \frac{2}{c} a_n \qquad \text{and} \qquad a_n < 2c\, b_n.$$

The ordinary comparison test then implies that $\sum_{n=1}^{\infty} b_n$ converges if $\sum_{n=1}^{\infty} a_n$ converges, and conversely.

7. Suppose $r < 1$. Choose s with $r < s < 1$. There is some N such that $\sqrt[n]{a_n} \leq s$ for $n \geq N$. Then $a_n \leq s^n$, so

$$\sum_{n=N}^{\infty} a_n \leq \sum_{n=N}^{\infty} s^n$$

converges. If $r > 1$, and $r > s > 1$, then there is some N such that $\sqrt[n]{a_n} \geq s$ for $n \geq N$. Thus $a_n \geq s^n \geq 1$, so $\sum_{n=1}^{\infty} a_n$ does not converge.

8. The sequence 1, -1, 1, -1, 1, -1,... is Cesaro summable to 1/2.

9. (a) Choose m so that $a_1,\ldots,a_n$ appear among $b_1,\ldots,b_m$.

(b) This follows immediately from part (a), since $\sum_{n=1}^{\infty} a_n$ is the least upper bound of all partial sums $s_n$. (Actually, we have only proved that the inequality holds if the sum $\sum_{n=1}^{\infty} b_n$ exists.)

(c) The reverse inequality $\sum_{n=1}^{\infty} b_n \leq \sum_{n=1}^{\infty} a_n$ follows from part (b), since $\{a_n\}$ is also a rearrangement of $\{b_n\}$. It follows that $\sum_{n=1}^{\infty} b_n$ exists and equals $\sum_{n=1}^{\infty} a_n$.

(d) Let $\{p_n\}$ and $\{q_n\}$ be the series formed of the positive and negative terms, respectively, of $\{a_n\}$, and let $\{p_n'\}$ and $\{q_n'\}$ be defined similarly for $\{b_n\}$. Then $\{p_n'\}$ is a rearrangement of $\{p_n\}$ and similarly for $\{q_n\}$. So by part (d), $\Sigma p_n' = \Sigma p_n$ and $\Sigma q_n' = \Sigma q_n$, the sums on the right existing because $\{a_n\}$ is absolutely convergent. Therefore, $\{b_n\}$ is absolutely convergent, and $\sum_{n=1}^{\infty} b_n = \Sigma p_n' - \Sigma q_n' = \Sigma p_n - \Sigma q_n = \sum_{n=1}^{\infty} a_n$.

10. (a) Let $b_j = a_{n_j}$. Then

$$\begin{aligned}|b_j| + |b_{j+1}| + \dots + |b_k| &= |a_{n_j}| + |a_{n_{j+1}}| + \dots + |a_{n_k}| \\ &\leq |a_{n_j}| + |a_{n_{j+1}}| + |a_{n_{j+2}}| + \dots + |a_{n_k}|.\end{aligned}$$

Consequently, $\lim_{j,k\to\infty} |b_j| + \dots + |b_k| = 0$.

(b) If $\sum_{n=1}^{\infty} a_n$ does not converge absolutely, then either $\Sigma p_n$ or $\Sigma q_n$ diverges, where $\Sigma p_n$ is the series of positive terms, and $\Sigma q_n$ the series of negative terms. Choose the appropriate one as $\Sigma b_n$.

(c) The series $a_1 + a_3 + a_5 + \ldots$ and $a_2 + a_4 + a_6 + \ldots$ both converge by part (a). The same is true of the series $a_1 + 0 + a_3 + 0 + a_5 + \ldots$ and $0 + a_2 + 0 + a_4 + 0 + a_6 + \ldots$, whose sum in $\sum_{n=1}^{\infty} a_n$.

11. For every N, we have

$$\left|\sum_{n=1}^{N} a_n\right| \leq \sum_{n=1}^{N} |a_n| \leq \sum_{n=1}^{\infty} |a_n|.$$

Since $\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n$ the result follows.

12. Choose $\delta > 0$ so that $|\sin x| \geq 1/2$ on $(k\pi + \pi/2 - \delta,\ k\pi + \pi/2 + \delta)$. Then

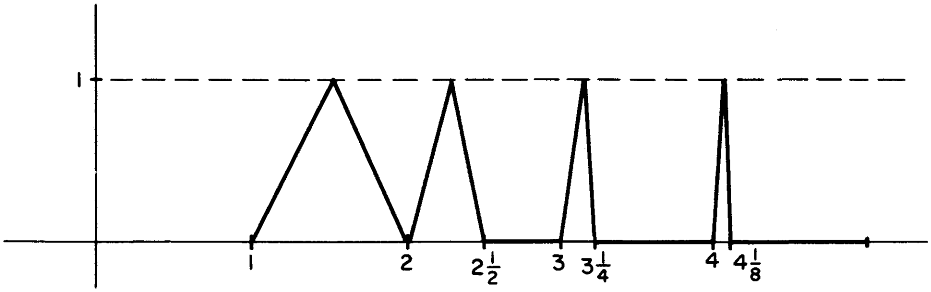
$$\int_{k\pi+\pi/2+\delta}^{k\pi+\pi/2-\delta} |(\sin x)/x| \, dx \geq \frac{\delta}{k\pi + \pi/2}.$$

Since the series

$$\sum_{k=1}^{\infty} \frac{1}{k\pi + \pi/2}$$

diverges, the same is true for the integral.

13. Let f be the function whose graph is shown below.



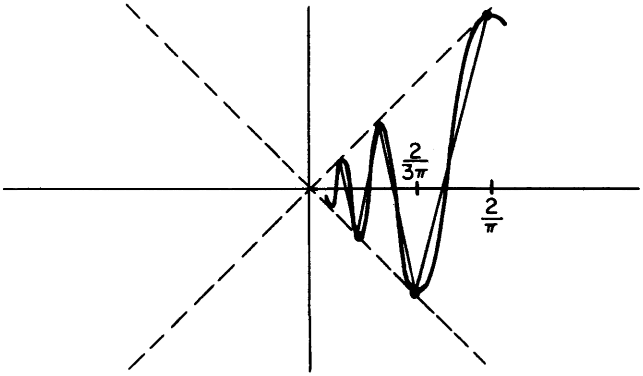
14. For the partition

$$P = \{0, \frac{2}{(2n+1)\pi}, \dots, \frac{2}{\pi}, 1\}$$

we have

$$\ell(f, P) > \frac{2}{\pi} + \frac{2}{3\pi} + \dots + \frac{2}{(2n+1)\pi} = \frac{2}{\pi}\left(1 + \frac{1}{3} + \dots + \frac{1}{2n+1}\right),$$

and these sums are not bounded.



16. (a)

$$\frac{\binom{\alpha}{k+1}r^{k+1}}{\binom{\alpha}{k}r^{k}} = \frac{r\alpha(\alpha - 1)\cdot\ldots\cdot(\alpha - k)/(k + 1)!}{(\alpha - 1)\cdot\ldots\cdot(\alpha - k + 1)/k!} = r\frac{\alpha - k}{k + 1}.$$

Clearly

$$\lim_{k\to\infty}\left|\frac{\alpha - k}{k + 1}\right| = 1,$$

so

$$\lim_{k\to\infty}\frac{\left|\binom{\alpha}{k+1}r^{k+1}\right|}{\left|\binom{\alpha}{k}r^{k}\right|} = |r|.$$

(b)

$$|R_{n,0}(x)| = \left|\binom{\alpha}{n+1}x^{n+1}(1 + t)^{\alpha-n-1}\right|$$

$$\leq \left|\binom{\alpha}{n+1}x^{n+1}\right| \to 0, \qquad \text{by part (a).}$$

(c) We have $0 < 1 + x < 1 + t \leq 1$. If $\alpha - 1 \geq 0$, then $(1 + t)^{\alpha-1} \leq 1$. If $\alpha - 1 < 0$, then $(1 + t)^{\alpha-1} < (1 + x)^{\alpha-1}$. So $(1 + t)^{\alpha-1} \leq M$, so $|x(1 + t)^{\alpha-1}| \leq |x|M$. Moreover, since $-1 < x$ and $t \leq 0$, we have

$$-t \geq xt,$$

$$0 > x - t \geq x + xt,$$

$$0 < \frac{x - t}{x} \leq 1 + t, \qquad \text{since } x < 0$$

$$0 < 1 - t/x \leq 1 + t,$$

$$0 < \frac{1 - t/x}{1 + t} \leq 1, \qquad \text{since } 1 + t > 0.$$

Thus

$$|R_{n,0}(x)| = \left|(n+1)\binom{\alpha}{n+1}x(1+t)^{\alpha-1}\left(\frac{x-t}{1+t}\right)^n\right|$$
$$\le |x\alpha M|\cdot\left|\binom{\alpha-1}{n}x^n\right| \to 0 \qquad \text{by part (a).}$$

17. (a) According to Problem 21-20(b), if $m \le a_1 + \dots + a_n \le M$, then

$$b_k m \le a_k b_k + \dots + a_n b_n \le b_k M.$$

Since $\lim_{k\to\infty} b_k = 0$, this shows that $\lim_{k,n\to\infty} a_k b_k + \dots + a_n b_n = 0$.

(b) Let $a_n = (-1)^{n+1}$; the partial sums are bounded. So if $b_1 \ge b_2 \ge b_3 \ge \dots \ge 0$ and $\lim_{n\to\infty} b_n = 0$, then $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ converges.

(c) Choose $b_n = 1/n$ and $a_n = \cos nx$. The partial sums of $\{a_n\}$ are bounded because, by Problem 15-31,

$$|\cos x + \dots + \cos nx| = \left|\frac{\sin(n+1/2)x}{2\sin x/2} - \frac{1}{2}\right| \le \left|\frac{\sin(n+1/2)x}{2\sin x/2}\right| + \frac{1}{2}$$
$$\le \frac{1}{2|\sin x/2|} + \frac{1}{2}.$$

18. Since

$$a_1 \ge a_2 \ge \underbrace{a_3 \ge a_4} \ge \underbrace{a_5 \ge a_6 \ge a_7 \ge a_8} \ge \dots,$$

we have

$$a_2 \le a_2$$

$$2a_4 \le a_3 + a_4$$

$$4a_8 \le a_5 + a_6 + a_7 + a_8,$$

etc.

So

$$\sum_{n=1}^{N} 2^n a_{2^n} \le 2\left(\sum_{k=1}^{2^N} a_k\right) \le 2 \sum_{k=1}^{\infty} a_k.$$

19. (a) This problem depends upon Problem 1-18, or more precisely, on the following generalization:

$$|x_1y_1 + \dots + x_ny_n| \le \sqrt{x_1^2 + \dots + x_n^2}\,\sqrt{y_1^2 + \dots + y_n^2}.$$

(This may be proved, for example, by imitating part (b) of Problem 1-18, or see the solution to Problem 13-33(c).) We have

$$|a_nb_n + \dots + a_mb_m| \le \sqrt{a_n^2 + \dots + a_m^2}\,\sqrt{b_n^2 + \dots + b_m^2}.$$

This shows that the Cauchy condition for $\{a_n^2\}$ and $\{b_n^2\}$ implies the Cauchy condition for $\{a_nb_n\}$.

(b) Apply part (a) with $b_n = 1/n$.

20. Choose n so that $a_n + \dots + a_m < \epsilon$ for $m \ge n$. Then

$$(m - n)a_m \le a_n + \dots + a_m < \epsilon.$$

Since $\lim_{m \to \infty} m/(m - n) = 1$ and

$$ma_m = \frac{m}{m - n} \cdot (m - n)\ a_n,$$

it follows that $ma_m < 2\epsilon$ for sufficiently large m.

21. Let $\{a_n\}$ be

$$1, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \ldots .$$

22. Suppose $p/q < 1/n$ and $p/q > 1/(n + 1)$. Then $np < q$, and

$$\frac{p}{q} - \frac{1}{n + 1} = \frac{pn + p - q}{q(n + 1)}.$$

The numerator $pn + p - q$ is $< q + p - q = p$. (Of course, it may be even smaller when the fraction is expressed in lowest terms.)

# CHAPTER 23

1. (ii) $f(x) = \lim_{n\to\infty} f_n(x) = 0$. The sequence $\{f_n\}$ converges uniformly to f (in fact, is eventually 0) on $[a, b]$, but does not converge uniformly to f on $\mathbf{R}$.

(iv)

$$f(x) = \lim_{n\to\infty} e^{-nx^2} = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0. \end{cases}$$

$\{f_n\}$ does not converge uniformly to f.

2. (ii) $\log(-a) - \frac{x}{a} - \frac{x^2}{2a^2} - \frac{x^3}{3a^3} - \frac{x^4}{4a^4} - \cdots$ .

(iv) $\sum_{k=0}^{\infty} \binom{-1/2}{k} x^{2k}$.

3. (ii) $1/(1 + x^3)$.

(iv) By Problem 10,

$$\frac{1}{2\cdot 1} - \frac{1}{3\cdot 2} + \frac{1}{4\cdot 3} - \frac{1}{5\cdot 4} = \lim_{x\to 1} \frac{x^2}{1\cdot 2} - \frac{x^2}{3\cdot 2} + \frac{x^4}{4\cdot 3} - \frac{x^5}{5\cdot 4} + \cdots$$

$$= \lim_{x\to 1} (1 + x)\log(1 + x) - x \qquad \text{by part (iii)}$$

$$= 2 \log 2 - 2.$$

(v) If

$$f(x) = x - \frac{x^4}{4} + \frac{x^7}{7} - \frac{x^{10}}{10} + \dots,$$

then for $|x| < 1$ we have

$$f'(x) = 1 - x^3 + x^6 - x^9 + \dots = \frac{1}{1 + x^3} \qquad \text{by part (ii)}$$

$$= \frac{1}{(x + 1)(x^2 - x + 1)} = \frac{1/3}{x + 1} + \frac{-x/3 + 2/3}{x^2 - x + 1},$$

$$= \frac{1/3}{x + 1} - \frac{1}{6}\frac{2x - 1}{x^2 - x + 1} + \frac{1}{2}\cdot\frac{1}{x^2 - x + 1},$$

so

$$f(x) = \frac{1}{3}\log(x + 1) - \frac{1}{6}\log(x^2 - x + 1) + \frac{3\sqrt{3}}{16}\arctan\frac{x - 1/2}{\sqrt{3/4}} + \frac{3\sqrt{3}}{16}\arctan\frac{1/2}{\sqrt{3/4}}$$

$$= \frac{1}{3}\log(x + 1) - \frac{1}{6}\log(x^2 - x + 1) + \frac{3\sqrt{3}}{16}\arctan\frac{x - 1/2}{\sqrt{3/4}} + \frac{3\sqrt{3}}{16}\,\frac{\pi}{6}.$$

Consequently,

$$1 - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \dots = \lim_{x \to 1} f(x) = \frac{\log 2}{3} + \frac{\sqrt{3}}{16}\,\pi.$$

5. (a)

$$(1+x)f'(x) = (1+x)\sum_{n=1}^{\infty} n\binom{\alpha}{n}x^{n-1}$$

$$= \sum_{n=0}^{\infty} \left[n\binom{\alpha}{n} + (n+1)\binom{\alpha}{n+1}\right]x^{n}$$

$$= \sum_{n=0}^{\infty} \alpha\binom{\alpha}{n}x^{n} = \alpha f(x).$$

(b)

$$g'(x) = \frac{(1+x)^{\alpha}f'(x) - f(x)\alpha(1+x)^{\alpha-1}}{(1+x)^{2\alpha}} = 0 \qquad \text{by part (a),}$$

so $g$ is a constant $c$. Thus $f(x) = c(1+x)^{\alpha}$. Since $f(0) = 1$, we have $c = 1$.

6. (a) $a_n = f^{(n)}(0)/n! = 0$.

(b) $a_0 = f(0) = \lim_{n\to\infty} f(x_n) = 0$, since $f$ is continuous at 0. Thus

$$f(x) = \sum_{n=1}^{\infty} a_n x^n = x\left(\sum_{n=1}^{\infty} a_n x^{n-1}\right) = x\left(\sum_{n=0}^{\infty} a_{n+1}x^{n}\right) = xg(x).$$

Now $g(x_n) = 0$ (for all $x_n \neq 0$), so by the result just proved, $a_1 = 0$. Thus

$$f(x) = x^2 \sum_{n=0}^{\infty} a_{n+2}x^n,$$

so $a_2 = 0$, etc.

(c) Apply part (b) to $f - g$.

7. If $f$ is even, then $f^{(n)}$ is odd for $n$ odd, so $a_n = f^{(n)}(0)/n! = 0$ for $n$ odd. If $f$ is odd, then $f^{(n)}$ is odd for $n$ even, so $a_n = 0$ for $n$ even.

8. (a) Clearly $a_{n-1} \leq a_n$. Hence

$$\frac{a_{n+1}}{a_n} \leq \frac{2a_n}{a_n} \leq 2.$$

(b)

$$\frac{|a_{n+1}x^n|}{|a_nx^{n-1}|} \leq 2|x| < 1 \quad \text{for } |x| < 1/2.$$

(c) We have

$$\begin{aligned}
f(x) &= \sum_{n=1}^{\infty} a_nx^{n-1} = 1 + x + 2x^2 + 3x^3 + \ldots,\\
xf(x) &= \sum_{n=1}^{\infty} a_nx^n = \phantom{1 +{}} x + x^2 + 2x^3 + \ldots,\\
x^2f(x) &= \sum_{n=1}^{\infty} a_nx^{n+1} = \phantom{1 + x +{}} x^2 + x^3 + \ldots,
\end{aligned}$$

so

$$f(x) = 1 + xf(x) + x^2f(x).$$

(d) Let $\alpha = (-1 - \sqrt{5})/2$ and $\beta = (-1 + \sqrt{5})/2$. Then

$$f(x) = \frac{-1}{x^2 + x - 1} = \frac{1/\sqrt{5}}{x - \left(\frac{-1 - \sqrt{5}}{2}\right)} + \frac{-1/\sqrt{5}}{x - \left(\frac{-1 + \sqrt{5}}{2}\right)} = \frac{1/\sqrt{5}}{x - \alpha} - \frac{1/\sqrt{5}}{x - \beta}$$

$$= \frac{1}{\sqrt{5}}\left(- \frac{1}{\alpha} - \frac{x}{\alpha^2} - \frac{x^2}{\alpha^3} - \dots\right) - \frac{1}{\sqrt{5}}\left(- \frac{1}{\beta} - \frac{x}{\beta^2} - \frac{x^2}{\beta^3} - \dots\right).$$

Consequently,

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1}{\beta^n} - \frac{1}{\alpha^n}\right)$$

$$= \frac{1}{\sqrt{5}}\left[\frac{1}{\left(\frac{-1 + \sqrt{5}}{2}\right)^n} - \frac{1}{\left(\frac{-1 - \sqrt{5}}{2}\right)^n}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{\left(\frac{-1 - \sqrt{5}}{2}\right)^n - \left(\frac{-1 + \sqrt{5}}{2}\right)^n}{\left(\frac{-1 - \sqrt{5}}{2}\right)^n \cdot \left(\frac{-1 + \sqrt{5}}{2}\right)^n}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{\left(\frac{-1 - \sqrt{5}}{2}\right)^n - \left(\frac{-1 + \sqrt{5}}{2}\right)^n}{\left(\frac{1 - 5}{4}\right)^n}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1 + \sqrt{5}}{2}\right)^n - \left(\frac{1 - \sqrt{5}}{2}\right)^n\right].$$

9. The power series for $f(x) = \log(1 - x)$ is $\sum_{n=0}^{\infty} (-1)^n a_n x^n$, where $\sum_{n=0}^{\infty} a_n x^n$ is the power series for $h(x) = \log(1 + x)$. Since $\sum_{n=0}^{\infty} a_n x^n$ converges only for $-1 < x \leq 1$, the power series $\sum_{n=0}^{\infty} (-1)^n a_n x^n$ converges only for $-1 \leq x < 1$. Since $g(x) = f(x) - h(x)$, its power series converges only for $-1 < x < 1$.

10. If $0 \leq x \leq 1$, then $x \geq x^2 \geq x^3 \geq \dots \geq 0$. Consequently Abel's Lemma shows that $|a_m x^m + \dots + a_n x^n| < \epsilon$ if $|a_m + \dots + a_n| < \epsilon$. The latter condition is true for sufficiently large m and n. Consequently, $|a_m x^m + a_{m+1} x^{m+1} + \dots$ $< \epsilon$ for sufficiently large m and all x in $[0, 1]$. This means that for all x in $[0, 1]$,

$$\left|\sum_{n=0}^{\infty} a_n x^n - (a_0 + \dots + a_{m-1} x^{m-1})\right| < \epsilon$$

for sufficiently large m. This is precisely the assertion that $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly in $[0, 1]$.

11. Let $a_n = (-1)^n$. For $0 \leq x < 1$, we have

$$\sum_{n=0}^{\infty} a_n x^n = 1 - x + x^2 - x^3 + \dots = \frac{1}{1 + x},$$

so

$$\lim_{x \to 1^-} \sum_{n=0}^{\infty} a_n x^n = \lim_{x \to 1^-} \frac{1}{x + 1} = \frac{1}{2}.$$

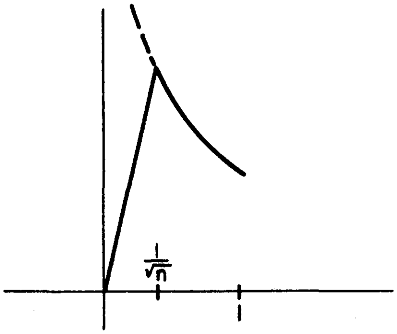
12. Given $\epsilon > 0$, choose N so that $|g(x) - f_n'(x)| < \epsilon$ for all x in $[a, b]$. Then

$$\left|\int_a^x g - \int_a^x f_n'\right| < \epsilon$$

for all x in $[a, b]$. But $\int_a^x g = f(x) - f(a)$ (see the proof of Theorem 3), so $|f(x) - f_n'(x)| < \epsilon$ for all x in $[a, b]$.

13. (a) Choose $N$ so that if $n \geq N$, then $|f(x) - f_n(x)| \leq 1$ for all $x$ in $[a, b]$. Since $f_N$ is bounded, there is some $M$ such that $|f_N(x)| \leq M$ for all $x$ in $[a, b]$. Then $|f(x)| \leq |f_N(x)| + 1 \leq M + 1$.

(b) Let $f_n(x) = nx$ for $0 \leq x \leq 1/\sqrt{n}$, and $f(x) = 1/x$ for $1/\sqrt{n} \leq x \leq 1$. Then $\lim\limits_{n \to \infty} f_n(0) = 0$ and $\lim\limits_{n \to \infty} f_n(x) = 1/x$ for $0 < x \leq 1$.



14. Let $f_n(x) = [f(x + 1/n) - f(x)]/(1/n)$.

15. Let $\{a_n\}$ be the sequence 0, 1, 1/2, 1/3, 2/3, 1/4, 3/4, 1/5, 2/5, 3/5, 4/5, 1/6, 5/6,... . Let $f_n(x) = 0$ if $x \neq a_1,\ldots,a_n$, and let $f_n(a_j) = 1$ for $1 \leq j \leq n$.

16. (a) Suppose $\{u_0,\ldots,u_m\}$ contains $\{t_0,\ldots,t_n\}$. For each $i$ we have

$$t_i = u_\alpha < u_{\alpha+1} < \ldots < u_\beta = t_{i+1}$$

for some $u_\alpha,\ldots,u_\beta$. Then $f$ has the constant value $s_i$ on each

$(u_{\alpha+j-1}, u_{\alpha+j})$. Thus the sum $\sum_{i=1}^{n} s_i(t_i - t_{i-1})$ is the sum $\sum_{j=1}^{m} s'_j(u_j - u_{j-1})$ where $s'_j$ is the constant value of $f$ on $(u_j - u_{j-1})$. To deal with the general case, consider a partition containing both $\{u_0,\ldots,u_m\}$ and $\{t_0,\ldots,t_n\}$.

(b) Choose $N$ so that for $n > N$ we have $|f(x) - s_n(x)| < \epsilon/2$ for all $x$ in $[a, b]$.

(c) From $|s_n(x) - s_m(x)| < \epsilon$ it follows easily that

$$\left|\int_a^b s_n - \int_a^b s_m\right| < \epsilon(b - a).$$

(d) Choose $N$ so that for $n > N$ we have both $|f(x) - s_n(x)| < \epsilon/2$ and $|f(x) - s_m(x)| < \epsilon/2$ for all $x$ in $[a, b]$.

(e) For any $\epsilon > 0$, choose $N$ so that if $n > N$, then

$$\left|\lim_{n\to\infty} \int_a^b s_n - \int_a^b s_n\right| < \frac{\epsilon}{3},$$

$$\left|\lim_{n\to\infty} \int_a^b t_n - \int_a^b t_n\right| < \frac{\epsilon}{3},$$

$$|s_n(x) - t_n(x)| < \frac{\epsilon}{3(b - a)} \qquad \text{for all } x \text{ in } [a, b].$$

The last equation implies that

$$\left|\int_a^b s_n - \int_a^b t_n\right| < \frac{\epsilon}{3}.$$

It follows that $\left|\lim_{n\to\infty} \int_a^b s_n - \lim_{n\to\infty} \int_a^b t_n\right| < \epsilon$.

(f) Let

$A = \{y\colon a \le y \le b$ and there is a step function $s$ on $[a, y]$ such that

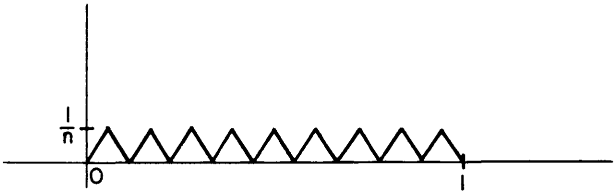
$|f(x) - s(x)| < \epsilon$ for all $x$ in $[a, y]\}$.

Let $\alpha = \sup A$. Since $f$ is continuous at $\alpha$, there is a $\delta > 0$ such that $|f(x) - f(\alpha)| < \epsilon$ for $|x - \alpha| < \delta$. There is some $y$ in $A$ with $\alpha - \delta < y < \alpha$. Thus there is a step function $s$ defined on $[a, y]$ with $|f(x) - s(x)| < \epsilon$ for all $x$ in $[a, y]$. Define $s_1(x) = s(x)$ for $x$ in $[a, y]$ and $s_1(x) = f(\alpha)$ for $y < x \le \alpha$. Then $s_1$ is a step function defined on $[a, \alpha]$ with $|f(x) - s_1(x)| < \epsilon$ for all $x$ in $[a, \alpha]$. This shows that $\alpha$ is in $A$. Similarly, if $\alpha < b$, then pick $\delta$ as before, and let $s$ be a step function defined on $[a, \alpha]$ with $|f(x) - s(x)| < \epsilon$ for $x$ in $[a, \alpha]$. If $s_1(x)$ is defined as $s(x)$ for $x$ in $[a, \alpha]$ and as $f(\alpha)$ for $\alpha < x \le \alpha + \delta/2$, then $|f(x) - s_1(x)| < \epsilon$ for $x$ in $[a, \alpha + \delta/2]$. So $\alpha + \delta/2$ is in $A$, contradicting the definition of $\alpha$. So $\alpha = b$, which completes the proof.

[The class of regulated functions can be determined more explicitly, as follows. A step function $s$ has the property that $\lim_{x\to a^+} s(x)$

and $\lim_{x \to a^-} s(x)$ exist for all a. It is not hard to show that a uniform limit of step functions must have the same property (the proof is a simple modification of the proof of Theorem 2.) The converse is also true--if f has right and left hand limits at every point, then f is regulated. Notice that the class of regulated functions is smaller than the class of integrable functions. For example, if $f(0) = 0$ and $f(x) = \sin 1/x$ for $0 < x \leq 1$, then f is integrable on [0, 1] (by Problem 13-14, for example), but is not regulated.]

17. The function $f_n$ is shown below. The length of each $f_n$ is 2, since two sides of an equilateral triangle have a total length of twice the other side.



## CHAPTER 24

1. (ii) $|(3 + 4i)^{-1}| = 1/|3 + 4i| = 1/5$; $\theta$ = -argument of $3 + 4i = -\arctan 4/3$.

(iv) $|\sqrt[7]{3 + 4i}| = \sqrt[7]{|3 + 4i|} = \sqrt[7]{5}$; $\theta = (\arctan 4/3)/7$.

2. (ii) $(x^2)^2 + x^2 + 1 = 0$, so

$$x^2 = \frac{-1 \pm \sqrt{1 - 4}}{2}$$

$$= \frac{-1 + \sqrt{3}i}{2} \text{ or } \frac{-1 - \sqrt{3}i}{2}$$

$$= \cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3} \text{ or } \cos\frac{4\pi}{3} + i \sin\frac{4\pi}{3}.$$

So x is one of the square roots of these numbers, so x is one of

$$\cos\frac{\pi}{3} + i \sin\frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i,$$

$$\cos\frac{4\pi}{3} + i \sin\frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i,$$

$$\cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i,$$

$$\cos\frac{5\pi}{3} + i \sin\frac{5\pi}{3} = \frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

(iv) $x = \frac{7}{3} - \frac{4i}{3}$,

$y = \frac{1}{3} + 2i$.

3. (ii) All $z$ with $|z| = 1$.

(iv) The ellipse consisting of all points the sum of whose distances from $a$ and $b$ is $c$, if $c > |a - b|$; the line segment between $a$ and $b$ if $|a - b| = c$; $\emptyset$ if $|a - b| > c$.

6. $z$ and $\sqrt{i}\cdot\overline{z\sqrt{-i}}$ are reflections of each other through the diagonal, because the diagonal goes into the real axis under multiplication by $\sqrt{-i}$, so $\sqrt{i}\cdot\overline{z\sqrt{-i}}$ is obtained by rotating the plane until the diagonal lies along the real axis, then reflecting through the real axis, and then rotating back by the same amount.

7. (a) Since $a_0,\ldots,a_{n-1}$ are real, we have

$$0 = \overline{(a + bi)^n + a_{n-1}(a + bi)^{n-1} + \ldots + a_0}$$

$$= (\overline{a + bi})^n + a_{n-1}(\overline{a + bi})^{n-1} + \ldots + a_0.$$

(b) Since $a + bi$ and $a - bi$ are roots, $z^n + a_{n-1}z^{n-1} + \ldots + a_0$ is divisible by $z - (a + bi)$ and $z - (a - bi)$, and by their product

$$[z - (a + bi)]\cdot[z - (a - bi)] = z^2 - 2az + (a^2 + b^2).$$

8. (a) Suppose that $a + b\sqrt{c} = a' + b'\sqrt{c}$. If $b = b'$, then also $a = a'$. If $b \neq b'$, then we would have $\sqrt{c} = (a - a')/(b - b')$, contradicting the fact that $\sqrt{c}$ is irrational (Problem 2-16).

(b) The proofs are almost exactly the same as parts (1)-(6) of Theorem 1.

(c) Since $a_0,\ldots,a_{n-1}$ are integers, we have

$$\begin{aligned} 0 &= \overline{(a + b\sqrt{c})^n + a_{n-1}(a + b\sqrt{c})^{n-1} + \cdots + a_0} \\ &= (\overline{a + b\sqrt{c}})^n + a_{n-1}(\overline{a + b\sqrt{c}})^{n-1} + \cdots + a_0. \end{aligned}$$

10. (a) If $\omega^n = 1$, then $(\omega^k)^n = \omega^{nk} = (\omega^n)^k = 1$.

(b) There are two primitive 3rd roots and 4 primitive 5th roots (in each case, all roots except 1); there are two primitive 4th roots ($i$ and $-i$) and six primitive 9th roots (if $\omega$ is the root with smallest argument, then 1, $\omega^3$, and $\omega^6$ are not primitive). [In general, the number of primitive $n$th roots is the number of numbers from 1 to $n - 1$ that have no factor in common with $n$.]

(c) By Problem 2-4,

$$1 + \omega + \cdots + \omega^{n-1} = \frac{1 - \omega^n}{1 - \omega} = 0.$$

11. (a) The assertion is clear if the line is the real axis, because in that case the imaginary parts of $z_1,\ldots,z_k$ are either all positive or all negative, so the same is true for the sum. In general, let $\theta$ be the angle between the line and the real axis, and let $w = \cos\theta + i\sin\theta$.

Then $z_1w^{-1},\ldots,z_kw^{-1}$ all lie on one side of the real axis, so the same is true of $z_1w^{-1},\ldots,z_kw^{-1} = (z_1 + \ldots + z_k)w^{-1}$, which shows that $z_1 + \ldots + z_k$ lies on the corresponding side of the original line.

(b) $z^{-1}$ is above the real axis if and only if $z$ is below the real axis, and conversely. This proves the assertion when the line is the real axis. The general case then follows as in part (a).

12. The hypotheses remain true when each $z_j$ is multiplied by the same $w$. So we can assume that $z_1$ is real, in fact, that $z_1 = 1$. It follows that $z_2 + z_3$ is real, so $z_2 = a + b_i$, $z_3 = a - b_i$. Moreover, $2a + 1 = 0$, so $a = -1/2$; since $a^2 + b^2 = 1$, we have $b = \sqrt{3}/2$. The points 1, $-1/2 + i\sqrt{3}/2$, and $-1/2 - i\sqrt{3}/2$ do lie on the vertices of an equilateral triangle.

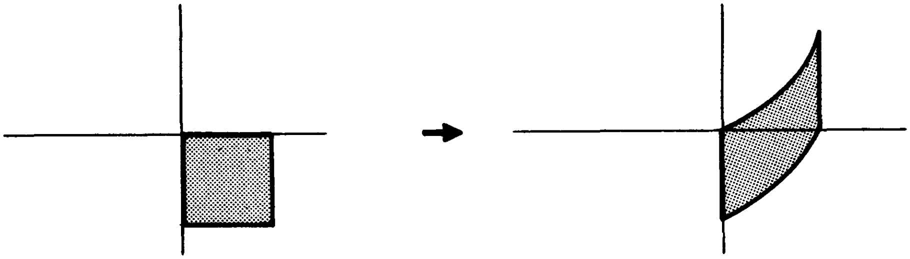
# CHAPTER 25

1. (a) If $|x - x_0| < \delta$, then $|\alpha(x) - \alpha(x_0)| < \delta$. Similarly, if $|y - y_0| < \delta$, then $|\beta(y) - \beta(y_0)| < \delta$.

(b) $g = f \circ \alpha$; $h = f \circ \beta$.

2. (a) $g$ is a continuous real-valued function on $[0, 1]$ with $g(0) = f(z)$ and $g(1) = f(w)$. So $g$ takes on all values between $f(z)$ and $f(w)$ on $[0, 1]$.

(b) Let $f(x + iy) = x + i(y + x^2)$, on $[0, 1] \times [-1, 0]$.



3. (a) There is, by the Fundamental Theorem of Algebra, some number $z_1$ such that $z_1{}^n + a_{n-1}z_1{}^{n-1} + \dots + a_0 = 0$. Then

$$z^n + a_{n-1}z^{n-1} + \dots + a_0 = (z - z_1)(z^{n-1} + b_{n-2}z^{n-2} + \dots + b_0)$$

for some numbers $b_0, \dots, b_{n-2}$ (as in Problem 3-7). Using an inductive argument, we can assume that

$$z^{n-1} + b_{n-2}z^{n-2} + \dots + b_0 = \prod_{i=2}^{n} (z - z_i)$$

for some numbers $z_2, \dots, z_n$.

(b) According to Problem 24-7, the non-real numbers $z_1, \dots, z_n$ from part (a) occur in pairs which are conjugates of each other, and $(z - z_i)(z - \overline{z}_i)$ has real coefficients.

4. (a) is obvious.

(b) If $f = \sum_{i=1}^{n} h_i^2$ and $g = \sum_{j=1}^{m} k_j^2$, then $fg = \sum_{i=1}^{n} \sum_{j=1}^{m} (h_i k_j)^2$.

(c) If $f(a) = 0$, then $f(x) = (x - a)f_1(x)$ for some polynomial function $f_1$. Then $f_1(x) \geq 0$ for $x \geq a$, and $f_1(x) \leq 0$ for $x \leq a$. So $f_1(a) = 0$. Thus every root of $f$ is a double root, so $f(x) = \prod_{i=1}^{k} (x - a_i)^2 g(x)$ where $g(x) > 0$ for all $x$. Since $g$ has no roots, Problem 3 shows that $g$ is a product of quadratic factors $x^2 + ax + b$ without roots. Thus $a^2 - 4b < 0$, so we can write

$$x^2 + ax + b = (x + \tfrac{a}{2})^2 + (\sqrt{b - a^2/4})^2,$$

which is a sum of squares. So $f$ is a product of sums of squares, so $f$ is a sum of squares.

5. (a) Follow the procedure given in the hint, to obtain a decreasing sequence of rectangles $[a_i, b_i] \times [c_i, d_i]$, each containing infinitely many points of $A$. By the Nested Intervals Theorem (Problem 8-14), there is a point $x$ in all $[a_i, b_i]$ and a point $y$ in $[c_i, d_i]$. Then

$z = (x, y) = x + iy$ is in all $[a_i, b_1]x[c_i, d_i]$. If $\epsilon > 0$, then for some i the set $[a_i, b_i]x[c_i, d_i]$ is contained in $\{a: |z - a| < \epsilon\}$, so there are infinitely many points of A in $\{a: |z - a| < \epsilon\}$.

(b) If f were not bounded on $[a, b]x[c, d]$, then there would be points $a_n$ in $[a, b]x[c, d]$ with $|f(a_n)| \geq N$. If z is a limit point of $\{a_n: n \text{ in } \mathbf{N}\}$, then for every $\epsilon > 0$ there are points $a_n$ with $|a_n - z| < \epsilon$, so $f(a_n) \geq N$. This contradicts the fact that f is continuous at z.

(c) Let $\alpha = \sup\{f(z): z \text{ in } [a, b]x[c, d]\}$; this exists by part (b). If $\alpha \neq f(z)$ for all z in $[a, b]x[c, d]$, then $g(z) = 1/(f(z) - \alpha)$ would be a continuous unbounded function on $[a, b]x[c, d]$.

6. (a) If $c = \alpha + \beta i$, then $z = a + bi$ satisfies $z^2 = c$ if and only if

$$a^2 - b^2 = \alpha,$$

$$2ab = \beta,$$

which can be solved to give

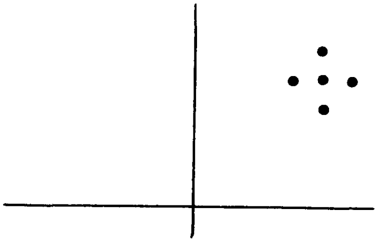
$$\left.\begin{aligned} a &= \sqrt{2\alpha + 2\sqrt{\alpha^2 + \beta^2}} \\ b &= \frac{\beta}{2\sqrt{2\alpha + 2\sqrt{\alpha^2 + \beta^2}}} \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} \alpha &= -\sqrt{2\alpha + 2\sqrt{\alpha^2 + \beta^2}} \\ \beta &= \frac{-\beta}{2\sqrt{2\alpha + 2\sqrt{\alpha^2 + \beta^2}}}. \end{aligned}\right.$$

(b) If $n = 2k$, then a solution of $z^k - \sqrt{c} = 0$ will be a solution of $z^{2n} - c = 0$. (If $k$ is even, we can continue until we reach an odd number.)

(c) For this $f$ we have

$$g(z) = f(z + z_0) = (z + z_0)^n - c = (z_0{}^n - c) + (nz_0)z + \dots .$$

(d) Suppose, for example, that $-c = \alpha + \beta i$ with $\alpha, \beta > 0$. If $\delta^n < \alpha$, then $|-c - \delta^n| < |-c|$. The same sort of argument works for all other cases.



7. (a)

$$\begin{aligned} f'(z) &= \sum_{\alpha=1}^{k} m_\alpha (z - z_1)^{m_1} \cdot \ldots \cdot (z - z_\alpha)^{m_\alpha - 1} \cdot \ldots \cdot (z - z_k)^{m_k} \\ &= \sum_{\alpha=1}^{k} m_\alpha (z - z_1)^{m_1} \cdot \ldots \cdot (z - z_\alpha)^{m_\alpha} \cdot \ldots \cdot (z - z_k)^{m_k} (z - z_\alpha)^{-1} \\ &= (z - z_1)^{m_1} \cdot \ldots \cdot (z - z_k)^{m_k} \cdot \sum_{\alpha=1}^{k} m_\alpha (z - z_\alpha)^{-1}. \end{aligned}$$

(b) If $z_1, \dots, z_k$ did all lie on the same side of a straight line through $z$, then $z - z_1, \dots, z - z_k$ would all lie on the same side of a straight line through 0. The same would then be true of $m_1^{-1}(z - z_1), \dots, m_k^{-1}(z - z_k)$, since $m_1, \dots, m_k > 0$. By Problem 24-11, this would imply that $g(z) \neq 0$, a contradiction.

(c) If $z$ satisfied $f'(z) = 0$ but $z$ were not in the convex hull of the set $\{z_1, \dots, z_k\}$, then there would be a straight line through $z$ containing the points $z_1, \dots, z_k$. This contradicts part (b).

8. The proof is exactly the same as for real-valued functions defined on $\mathbf{R}$.

9. (a) Let $z_0 = x_0 + iy_0$. Since

$$\alpha + i\beta = f'(z_0) = \lim_{z \to 0} \frac{f(z_0 + z) - f(z_0)}{z},$$

it must be true, in particular, that for real $\delta$ we have

$$\alpha + i\beta = \lim_{\delta \to 0} \frac{f(z_0 + \delta) - f(z_0)}{\delta} = \lim_{\delta \to 0} \left[ \frac{g(x_0 + \delta) - g(x_0)}{\delta} + i\, \frac{h(x_0 + \delta) - \delta(x_0)}{\delta} \right]$$

$$= g'(x_0) + ih'(x_0),$$

so $\alpha = g'(x_0)$ and $\beta = h'(x_0)$.

(b) We also have

$$\alpha + i\beta = \lim_{\delta \to 0} \frac{f(z_0 + \delta i) - f(z_0)}{\delta i} = \lim_{\delta \to 0} \left[ \frac{k(y_0 + \delta) - k(y_0)}{\delta i} + i\,\frac{\ell(y_0 + \delta) - \ell(y_0)}{\delta i} \right]$$

$$= \frac{k'(y_0)}{i} + \ell'(y_0),$$

so $\ell'(y_0) = \alpha$ and $k'(y_0) = -\beta$.

(c) Part (b) shows that u and v are constant along horizontal and vertical lines.

10. (a)

$$f^{(k)}(x) = \frac{1}{2i}\left( \frac{(-1)^k\, k!}{(x - i)^{k+1}} - \frac{(-1)^k\, k!}{(x + i)^{k+1}} \right).$$

(b)

$$\arctan^{(k)}(0) = f^{(k-1)}(0)$$

$$= \frac{(-1)^{k-1}(k - 1)!}{2i} \left( \frac{1}{(-i)^k} - \frac{1}{i^k} \right)$$

$$= \frac{(-1)^{k-1}(k - 1)!}{2i}\, i^k\, [1 + (-1)^{k-1}].$$

[If k is even, then $\arctan^{(k)}(0) = 0$. If $k = 2\ell + 1$, then $\arctan^{(2\ell+1)}(0) = (2\ell)!(-1)^{\ell}$.]

# CHAPTER 26

1. (ii) Absolutely convergent.

(iv) Absolutely convergent, since $|1/2 + i/2| = \sqrt{2}/2 < 1$.

2. (ii) The limit

$$\lim_{n\to\infty} \frac{|z|^{n+1}/(n+1)}{|z|^n/n} = |z| \lim_{n\to\infty} \frac{n}{n+1} = |z|$$

is $< 1$ for $|z| < 1$ and $> 1$ for $|z| > 1$.

(iv) The limit

$$\lim_{n\to\infty} \frac{|z|^{n+1}(n+1+2^{-n-1})}{|z|^n(n+2^{-n})} = |z| \lim_{n\to\infty} \frac{n+1+2^{-n-1}}{n+2^{-n}} = |z|$$

is $< 1$ for $|z| < 1$ and $> 1$ for $|z| > 1$.

3. (ii) Since

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}\,|z|}{\sqrt[n]{n^n}} = |z| \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{|z|}{e} \qquad \text{(by Problem 21-10)},$$

the radius of convergence is $e$.

(iv) Since

$$\lim_{n\to\infty} \frac{\sqrt[n]{n^2}\,|z|}{\sqrt[n]{2^n}} = |z| \lim_{n\to\infty} \frac{(\sqrt[n]{n})^2}{2} = \frac{|z|}{2} \qquad \text{(by Problem 21-1)},$$

the radius of convergence is 2.

4. (a) Since $\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|}\;|z| = \overline{\lim_{n\to\infty}} \sqrt[n]{|a_nz^n|}$, Problem 22-7 shows that $\sum_{n=0}^{\infty} a_nz^n$ converges (absolutely).

(b) If $\overline{\lim_{n\to\infty}} \sqrt[n]{|a_nz^n|} = 1 + \epsilon$ for $\epsilon > 0$, then there are infinitely many n with $\sqrt[n]{|a_nz^n|} \geq 1 + \epsilon/2$, so $|a_nz^n| \geq (1 + \epsilon/2)^n$ for infinitely many n, so the terms $a_nz^n$ are unbounded.

(c) Since the terms $\sqrt[n]{|a_n|}$ are unbounded, the same is true for $\sqrt[n]{|a_nz^n|}$ for $z \neq 0$. This is all the more true for $|a_nz^n|$, so $\sum_{n=0}^{\infty} a_nz^n$ diverges.

5. If z is on the unit circle, then $|z^n/n^2| \leq 1/n^2$, so $\sum_{n=1}^{\infty} |z_n|/n^2$ converges by the comparison test.

The series $\sum_{n=1}^{\infty} z^n$ certainly diverges for $z = 1$. If $z \neq 1$, then by Problem 2-4

$$\sum_{n=1}^{N} z^n = z\cdot\frac{1 - z^N}{1 - z}.$$

If $\lim_{N\to\infty} \sum_{n=1}^{N} z^n$ existed, then $\lim_{N\to\infty} z^N$ would exist, which is impossible, since $\ell = \lim_{N\to\infty} z^N$ would imply that $z\ell = \lim_{N\to\infty} z^{N+1} = \ell$, or $z = 1$.

The series $\sum_{n=1}^{\infty} z^n/n$ diverges for $z = 1$ and converges for $z = -1$.

6. (a) Clearly $e^0 = 1$. Now the function $g(z) = e^ze^{-z}$ satisfies $g'(z) = e^ze^{-z} - e^ze^{-z} = 0$, and $g(0) = 1$, so $e^ze^{-z} = 1$ for all z (Problem 25-9). In particular, $e^z \neq 0$ for all z. Now let $g(w) = e^{z+w}/e^w$. Then

$$g'(w) = \frac{e^w e^{z+w} - e^{z+w}e^w}{e^{2w}} = 0,$$

and $g(0) = e^z$, so $g(w) = e^z$ for all $w$, so $e^{z+w} = e^x e^z$ for all $w$.

(b)

$$\sin z \cos w + \cos z \sin w = \left(\frac{e^{iz} - e^{-iz}}{2i}\right)\left(\frac{e^{iw} + e^{-iw}}{2}\right) + \left(\frac{e^{iz} + e^{-iz}}{2}\right)\left(\frac{e^{iw} - e^{-iw}}{2i}\right)$$

$$= \frac{e^{i(z+w)} - e^{-i(z+w)}}{2i} = \sin(z + w).$$

$$\cos z \cos w - \sin z \sin w = \left(\frac{e^{iz} + e^{-iz}}{2}\right)\left(\frac{e^{iw} + e^{-iw}}{2}\right) - \left(\frac{e^{iz} - e^{-iz}}{2i}\right)\left(\frac{e^{iw} - e^{-iw}}{2i}\right)$$

$$= \frac{e^{i(z+w)} + e^{-i(z+w)}}{2} = \cos(z + w).$$

7. (a) Since $e^{iy} = \cos y + i \sin y$, the problem is just a restatement of Problem 15-21.

(b)

$$|e^{x+iy}| = |e^x \cdot e^{iy}| = |e^x| \cdot |\cos y + i \sin y| = |e^x|.$$

8. (a) If $z \neq 0$, then $z = r(\cos \theta + i \sin \theta)$ for some $r > 0$. Then $\exp(\log r + i\theta) = z$.

(b) We have $\sin z = w$ when

$$\frac{e^{iz} - e^{-iz}}{2i} = w,$$

$$(e^{iz})^2 - 2iwe^{iz} - 1 = 0,$$

$$e^{iz} = \frac{2iw \pm \sqrt{-4w + 4}}{2}$$

$$= iw \pm \sqrt{1 - w}.$$

This equation can always be solved for z, by part (a), since $iw \pm \sqrt{1 - w} \neq 0$.

9. (a) If $g(x) = u(x)$ and $h(x) = v(x)$ for real x, then Problem 25-9 shows that $f'(x) = g'(x) + ih'(x)$. Thus if we write $f' = u_1 + iv_1$ for real-valued $u_1$ and $v_1$, and define $g_1(x) = u_1(x)$ and $h_1(x) = v_1(x)$ for real x, then $g_1 = g'$ and $h_1 = h'$, so another application of Problem 25-9 shows that $f''(x) = g''(x) + ih''(x)$ for real x. Continuing inductively, we have $f^{(k)}(x) = g^{(k)}(x) + ih^{(k)}(x)$ for all k. Separating out the real and imaginary parts of

$$f^{(n)}(x) + a_{n-1}f^{(n-1)}(x) + \dots + a_0f(x) = 0 \qquad \text{for real } x$$

show that g and h satisfy (*).

(b) If $a = b + ci$ is a root of $z^n + a_{n-1}z^{n-1} + \dots + a_0 = 0$, then as in Problem 17-31, the function $f(z) = e^{az} = e^{bz} \cos cz + ie^{bz} \sin cz$ satisfies (*). So by part (a), the functions $g(x) = e^{bx} \cos cx$ and $h(x) = e^{bx} \sin cx$ satisfy (*).

10. (a), (b) $e^z = e^{x+iy} = e^x(\cos y + i \sin y) = w$ means that $e^x = |w|$, so $x = \log |w|$, and $y$ is an argument of $w$. In particular $e^{x+iy_0} = e^{x+iy_1}$ if $y_0$ and $y_1$ are arguments for $w$ (for example, $e^0 = e^{2\pi i}$), so exp is not one-one.

(c) Suppose that there was a continuous function log defined for $|z| = 1$ such that $\exp(\log (z)) = z$ for all $z$ with $|z| = 1$. Then we could write $\log(z) = \alpha(z) + i\beta(z)$ for continuous real-valued functions $\alpha$ and $\beta$. We must have $\alpha(z) = 0$ for $|z| = 1$, and $\beta(z)$ an argument of $z$ for $|z| = 1$. This contradicts the fact that there is no continuous argument function.

(d) All numbers of the form $i(2k\pi + \pi/2)$.

(e) All numbers of the form $e^{-(2k\pi+\pi/2)}$.

(f) $1^i$ has the values $e^{i(2k\pi i)} = e^{-2k\pi}$. The logarithms of these (real) numbers have the values $-2k\pi + 2\ell\pi i$. So $(1^i)^i$ has the values

$$e^{i(-2k\pi+2\ell\pi i)} = e^{-2\ell\pi}.$$

But $1^{-1}$ has only the values $e^{-(2k\pi i)} = 1$.

11. (a) $|x + i| = 1 + x^2$, and an argument for $x + 1$ is $\arctan 1/x$ $= \pi/2 - \arctan x$, while an argument for $x - i$ is $\arctan(-1/x)$ $= -\arctan 1/x = -(\pi/2 - \arctan x)$.

(b) From part (a) we obtain

$$\frac{1}{2i}[\log(x - i) - \log(x + i)] = \frac{1}{2i} \cdot - 2i(\frac{\pi}{2} - \arctan x)$$

$$= \arctan x - \pi/2,$$

which differs by a constant from the usual answer, arctan x.

2. (a) Since $a_n - a_m = (b_n - b_m) + i(c_n - c_m)$, we have $|b_n - b_m| \leq |a_n - a_m|$ and $|c_n - c_m| \leq |a_n - a_m|$, so $\{b_n\}$ and $\{c_n\}$ are Cauchy if $\{a_n\}$ is. Since we also have $|a_n - a_m| \leq |b_n - b_m| + |c_n - c_m|$, it follows that $\{a_n\}$ is Cauchy if $\{b_n\}$ and $\{c_n\}$ are.

(b) If $\{a_n\}$ is Cauchy, then $\{b_n\}$ and $\{c_n\}$ are, so $\{b_n\}$ and $\{c_n\}$ converge to $\alpha$ and $\beta$, respectively. Thus $\{a_n\}$ converges to $\alpha + i\beta$, by Theorem 1.

(c) The hint is the answer. Since Cauchy sequences of complex numbers are the same as convergent sequences of complex numbers, there is a Cauchy criterion for convergence of complex series: $\sum_{n=1}^{\infty} a_n$ converges if and only if $\lim_{m,n\to\infty} |a_{n+1} + \dots + a_m| = 0$. Now write down the proofs for the first halves of both Theorems 22-4 and 22-7, interpreting all numbers as complex numbers.

13.

$$\frac{1}{2} + \cos\theta + \ldots + \cos n\theta = \frac{1}{2}(e^{-in\theta} + \ldots + 1 + \ldots + e^{in\theta})$$

$$= \frac{e^{-in\theta}}{2}\,(1 + e^{i\theta} + \ldots + e^{2ni\theta})$$

$$= \frac{e^{-in\theta}}{2}\cdot\frac{1 - e^{i(2n+1)\theta}}{1 - e^{i\theta}}$$

$$= \frac{e^{-in\theta} - e^{i(n+1)\theta}}{2(1 - e^{i\theta})}$$

$$= \frac{e^{-1(n+1/2)\theta} - e^{i(n+1/2\theta)}}{2(e^{-i\theta/2} - e^{i\theta/2})}$$

$$= \frac{\sin(n + 1/2)\theta}{2\sin\theta/2}.$$

14. (a)

$$r_{n+1} = \frac{a_{n+2}}{a_{n+1}} = \frac{a_{n+1} + a_n}{a_{n+1}} = 1 + \frac{a_n}{a_{n+1}} = 1 + \frac{1}{r_n}.$$

(b) If $r = \lim\limits_{n\to\infty} r_n$ exists, then

$$r = \lim_{n\to\infty} r_n = \lim_{n\to\infty} 1 + \frac{1}{r_n} = 1 + \frac{1}{\lim\limits_{n\to\infty} r_n} = 1 + \frac{1}{r},$$

so $r = (1 + \sqrt{5})/2$ (clearly $r > 0$). To prove that the limit actually exists, note that if $r_n < (1 + \sqrt{5})/2$, then $r_n^2 - r_n - 1 < 0$, so

$$r_n < \frac{2r_n + 1}{r_n + 1} = r_{n+2}.$$

Thus $r_1 < r_3 < r_5 < \ldots < 2$, so $\lim\limits_{n\to\infty} r_{2n+1}$ exists. Similarly, $\lim\limits_{n\to\infty} r_{2n}$ exists. Moreover, the equation $r_{n+2} = (2r_n + 1)/(r_n + 1)$ leads, as before, to the fact that both limits are $(1 + \sqrt{5})/2$.

(c) The limit

$$\lim_{n\to\infty} \frac{|a_{n+1}z^{n+1}|}{|a_n z^n|} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}|z| = \frac{1+\sqrt{5}}{2}|z|$$

is $< 1$ for $|z| < 2/(1+\sqrt{5})$ and $> 1$ for $|z| > 2/(1+\sqrt{5})$.

15. (a)

$$-\frac{z}{2} + \frac{z}{2}\cdot\frac{e^z+1}{e^z-1} = \frac{z}{2}\left(-1 + \frac{e^z+1}{e^z-1}\right)$$

$$= \frac{z}{2}\left(\frac{-(e^z-1)+e^z+1}{e^z-1}\right)$$

$$= \frac{z}{e^z-1};$$

$$\frac{e^{-z}+1}{e^{-z}-1} = \frac{(e^{-z}+1)e^z}{(e^{-z}-1)e^z} = \frac{1+e^z}{1-e^z} = -\frac{e^z+1}{e^z-1}.$$

These formulas show that $z/(e^z - 1) = -z/2 + h(z)$, where $h$ is even. Consequently, the power series for $h$ contains only even powers of $z$. Thus $-1/2 = b_1$ is the coefficient of $z$ in the power series for $z/(e^z - 1)$, and $b_n = 0$ for odd $n > 1$.

(b) If $n > 1$, then the coefficient of $z^n$ must be 0. But this coefficient is $\sum_{i=0}^{n-1}\binom{n}{i}b_i$.

(c)

$$z \cot z = z\,\frac{\cos z}{\sin z} = z\cdot\frac{(e^{iz} + e^{-iz})/2}{(e^{iz} - e^{-iz})/2i}$$

$$= \frac{-2iz}{2}\cdot\frac{e^{2iz/2} + e^{-2iz/2}}{e^{2iz/2} - e^{-2iz/2}}$$

$$= \sum_{n=1}^{\infty} \frac{b_{2n}}{(2n)!}\,(2iz)^{2n}$$

$$= \sum_{n=1}^{\infty} \frac{b_{2n}}{(2n)!}\,(-1)^n 2^{2n} z^{2n}.$$

(d) From the formula $\tan 2z = 2 \tan z/(1 - \tan^2 z)$ (Problem 15-8) we have

$$\cot z - 2 \cot 2z = \frac{1}{\tan z} - \frac{1 - \tan^2 z}{\tan z} = \frac{\tan^2 z}{\tan z} = \tan z.$$

(e)

$$\tan z = \cot z - 2 \cot 2z$$

$$= \sum_{n=1}^{\infty} \frac{b_{2n}}{(2n)!}\,(-1)^n\, 2^{2n} z^{2n-1} - \sum_{n=1}^{\infty} \frac{b_{2n}}{(2n)!}\,(-1)^n\, 2^{2n} z^{2n-1}$$

$$= \sum_{n=1}^{\infty} \frac{b_{2n}}{(2n)!}\,(-1)^{n-1}\, 2^{2n}(2^{2n} - 1) z^{2n-1}.$$

16. (a) Applying (*) to $f^{(k)}$ we obtain

$$f^{(k)}(x+1) - f^{(k)}(x) = \sum_{n=1}^{\infty} \frac{f^{(k+n)}(x)}{n!}.$$

Thus

$$\sum_{k=0}^{\infty} \frac{b_k}{k!}[f^{(k)}(x+1) - f^{(k)}(x)] = \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \frac{b_k}{k!} \frac{f^{(k+n)}(x)}{n!}.$$

The coefficient of $f^{(1)}(x)$ is $b_0/0!1! = 1$. The coefficient of $f^{(j)}(x)$ for $j > 1$ is

$$\sum_{k=0}^{j-1} \frac{b_k}{k!(j-k)!} = \frac{1}{j!} \sum_{k=0}^{j-1} \binom{j}{k} b_k = 0, \qquad \text{by Problem 15(b).}$$

(b)

$$\begin{aligned} f'(0) + \dots + f'(n) &= \sum_{x=0}^{n} \left( \sum_{k=0}^{\infty} \frac{b_k}{k!}[f^{(k)}(x+1) - f^{(k)}(x)] \right) \\ &= \sum_{k=0}^{\infty} \frac{b_k}{k!} \sum_{x=0}^{n} [f^{(k)}(x+1) - f^{(k)}(x)] \\ &= \sum_{k=0}^{\infty} \frac{b_k}{k!} [f^{(k)}(n+1) - f^{(k)}(0)]. \end{aligned}$$

(c) Let $f$ be a polynomial function with $f' = g$. Then $f(n+1) - f(0) = \int_0^{n+1} g$. Since $b_0 = 1$, part (b) becomes

$$g(0) + \dots + g(n) = \int_0^{n+1} g(t)\,dt + \sum_{k=1}^{\infty} \frac{b_k}{k!} [g^{(k-1)}(n+1) - g^{(k-1)}(0)].$$

(d) $g^{(k)}(x) = p!\, x^{p-k}/(p-k)!$ for $k \le p$, so part (c) applied with $n - 1$ instead of $n$ gives

$$\sum_{k=1}^{n-1} k^p = \int_0^{n+1} x^p\,dx + \sum_{k=1}^{p+1} \frac{b_k}{k!}\,\frac{p!}{(p-k+1)!}\,n^{p-k+1}$$

$$= \frac{n^{p+1}}{p+1} + \sum_{k=1}^{p+1} \frac{b_k}{k}\binom{p}{k-1} n^{p-k+1}.$$

Thus

$$\sum_{k=1}^{n} k^p = \frac{n^{p+1}}{p+1} + n^p + b_1 n^p + \sum_{k=1}^{p+1} \frac{b_k}{k}\binom{p}{k-1} n^{p-k+1}$$

$$= \frac{n^{p+1}}{p+1} + \frac{n^p}{2} + \sum_{k=1}^{p+1} \frac{b_k}{k}\binom{p}{k-1} n^{p-k+1}.$$

17. (a) Clearly $\phi_n(0) = b_n$. If $n > 1$, then

$$\phi_n(1) = \sum_{k=0}^{n} \binom{n}{k} b_{n-k} = \sum_{k=0}^{n} \binom{n}{k} b_k, \qquad \text{since } \binom{n}{k} = \binom{n}{n-k}$$

$$= \sum_{k=0}^{n-1} \binom{n}{k} b_k + b_n = b_n, \quad \text{by Problem 15(b).}$$

$$\phi_n'(x) = \sum_{k=1}^{n} k\binom{n}{k} b_{n-k}x^{k-1}$$

$$= \sum_{k=0}^{n-1} (k+1)\binom{n}{k+1} b_{n-1+k}x^k$$

$$= \sum_{k=0}^{n-1} n\binom{n-1}{k} b_{n-1+k}x^k$$

$$= n\phi_{n-1}(x).$$

To prove the last equation, note first that

$$\phi_2(1 - x) = (1 - x)^2 - (1 - x) + \frac{1}{6} = x^2 - 2x + 1 - 1 + x + \frac{1}{6}$$

$$= x^2 - x + \frac{1}{6} = \phi_2(x).$$

Now suppose that $\phi_n(x) = (-1)^n\phi_n(1 - x)$ for some $n > 1$. Then the function $g(x) = \phi_{n+1}(1 - x)$ satisfies

$$g'(x) = -\phi_{n+1}'(1 - x) = -(n + 1)\phi_n(1 - x)$$

$$= (-1)^{n+1}(n + 1)\phi_n(x) = (-1)^{n+1}\phi_n'(x).$$

Moreover, $g(0) = \phi_{n+1}(1) = b_{n+1} = \phi_{n+1}(0)$, so $g(x) = (-1)^{n+1}\phi_{n+1}(x)$ for all x.

(b) Substituting from (*) we have

$$\sum_{k=0}^{N} \frac{b_k}{k!} [f^{(k)}(x + 1) - f^{(k)}(x)]$$

$$= \sum_{k=0}^{N} \frac{b_k}{k!} \sum_{n=1}^{N-k} \frac{f^{(k+n)}(x)}{n!} + \sum_{k=0}^{N} \frac{b_k}{k!} R_{N-k}{}^{k}(x).$$

The coefficient of $f'(x)$ in the double sum if $b_0/0!0! = 1$. For $1 < j \leq N$, the coefficient of $f^{(j)}(x)$ is

$$\sum_{k=0}^{j-1} \frac{b_k}{k!(j - k)!} = 0 \qquad \text{by Problem 15(b).}$$

(c) The term $R_{N-k}{}^{k}(x)$ is the remainder $R_{N-k,x}(x + 1)$ for the function $f^{(k)}$. Thus

$$R_{N-k}{}^{k}(x) = \int_{x}^{x+1} \frac{f^{(k+N-k+1)}(t)}{(N-k)!}(x+1-t)^{n}\,dt.$$

So

$$\sum_{k=0}^{N} \frac{b_k}{k!} R_{N-k}{}^{k}(x) = \int_{x}^{x+1} \sum_{k=0}^{N} \frac{b_k}{k!(N-k)!}(x+1-t)^{n} f^{(N+1)}(t)\,dt$$
$$= \int_{x}^{x+1} \frac{\phi_N(x+1-t)}{N!} f^{(N+1)}(t)\,dt.$$

(d) From parts (b) and (c) we obtain

$$f'(x) + \dots + f'(x+n) = \sum_{k=0}^{N} \frac{b_k}{k!}[f^{(k)}(x+n+1) - f^{(k)}(x)]$$
$$- \sum_{j=0}^{n} \int_{x+j}^{x+j+1} \frac{\phi_N(x+j+1-t)}{N!} f^{(N+1)}(t)\,dt.$$

Applying this to $g = f'$ we obtain the desired formula.

(e) If $t$ is in $[x+j, x+j+1]$, then $x-t$ is in $[-j-1, -j]$. Therefore, by definition, $\psi_N(x-t) = \phi_N(x+j+1-t)$.

18. (a) Apply Problem 17(d) to $g = \log$ and $x = 1$, with $n - 2$ for $n$. Since $g^{2}(x) = -1/x^{2}$ we obtain

$$\log(n-1)! = \log 1 + \ldots + \log(n-1)$$

$$= \int_1^n \log t\,dt + \frac{b_1}{1!}[\log n - \log 1] + (-1)^3 \int_1^n \frac{\psi_2(t)}{2!} \cdot - \frac{1}{t^2}\,dt$$

$$= \int_1^n \log t\,dt - \frac{1}{2}\log n + \int_1^n \frac{\psi_2(t)}{2t^2}\,dt$$

$$= n \log n - n + 1 - \frac{1}{2}\log n + \int_1^n \frac{\psi_2(t)}{2t^2}\,dt.$$

(b) Consequently

$$\log n! = \log(n-1)! + \log n = \left(n + \frac{1}{2}\right)\log n + 1 + \int_1^n \frac{\psi_2(t)}{2t^2}\,dt.$$

So

$$\log\left(\frac{n!}{n^{n+1/2}e^{-n}}\right) = 1 + \int_1^n \frac{\psi_2(t)}{2t^2}\,dt.$$

(c) Since $\psi_2$ is periodic, it is bounded. Thus $\int_1^\infty \psi_2(t)/2t^2\,dt$ exists since $\int_1^\infty 1/t^2\,dt$ exists. Let $\beta = 1 + \int_1^\infty \psi_2(t)/2t^2\,dt$. Then

$$\log\left(\frac{n!}{n^{n+1/2}e^{-n}}\right) = \beta - \int_n^\infty \frac{\psi_2(t)}{2t^2}\,dt$$

$$= \log \alpha - \int_n^\infty \frac{\psi_2(t)}{2t^2}\,dt,$$

so

$$\log\left(\frac{n!}{\alpha n^{n+1/2}e^{-n}}\right) = -\int_n^\infty \frac{\psi_2(t)}{2t^2}\,dt.$$

(d) Part (c) implies that

$$0 = \lim_{n\to\infty} \int_n^\infty \frac{\psi_2(t)}{2t^2}\,dt = \lim_{n\to\infty} \log\left(\frac{n!}{\alpha n^{n+1/2}e^{-n}}\right),$$

so

$$\lim_{n\to\infty} \frac{n!}{\alpha n^{n+1/2}e^{-n}} = 1.$$

Thus

$$\lim_{n\to\infty} \frac{(n!)^2}{\alpha^2 n^{2n+1}e^{-2n}} = 1,$$

$$\lim_{n\to\infty} \frac{(2n)!}{\alpha(2n)^{2n+1/2}e^{-2n}} = 1,$$

so

$$\begin{aligned}\sqrt{\pi} &= \lim_{n\to\infty} \frac{(n!)^2}{(2n)!}\cdot\frac{2^{2n}}{\sqrt{n}} \\ &= \lim_{n\to\infty} \frac{\alpha^2 n^{2n+1}e^{-2n}2^{2n}}{\alpha(2n)^{2n+1/2}e^{-2n}\sqrt{n}} \\ &= \alpha \lim_{n\to\infty} \frac{n^{2n+1/2}\sqrt{n}\;2^{2n}}{2^{2n}\sqrt{2}\;n^{2n+1/2}\sqrt{n}} \\ &= \frac{\alpha}{\sqrt{2}}.\end{aligned}$$

(e) $\phi_2(0) = \phi_2(1) = b_2 = 1/6$. If $0 = \phi_2'(x) = 2x - 1$, then $x = 1/2$, and $\phi_2(1/2) = 1/12$. So $1/12 \le |\phi_2(x)| \le 1/6$ for x in [0, 1]. From part (c) we have

$$-\frac{1}{12n} \le \log\left(\frac{n!}{\sqrt{2\pi}\, n^{n+1/2}e^{-n}}\right) \le \frac{1}{12n},$$

which yields the desired result.

# CHAPTER 27

1. It is clear that $a + b = b + a$, since the table for + is symmetric. Clearly $a + 0 = a$ for all a, and condition 2(ii) is satisfied because each row in the table contains 0. To check that $(a + b) + c = a + (b + c)$ it suffices to consider only cases where $a, b, c \neq 0$. Because $x + y = y + x$, this equation clearly holds when $a = c$. This leaves the cases

$$(1 + b) + 2 = 1 + (b + 2),$$
$$(2 + b) + 1 = 2 + (b + 1),$$

which are equivalent to each other, either of which can be checked by letting $b = 1$ and $2$. Conditions (4)-(6) are checked similarly. Finally, (7) is clear if $a = 0$ or $1$. For $a = 2$ we can assume $b, c \neq 0$ and the condition is clear if $b = c = 1$. This leaves only the cases $a = 2$, $b = 2$, $c = 2$ and $a = 2$, $b = 1$, $c = 2$ and $a = 2$, $b = 2$, $c = 1$, the last two being equivalent.

F cannot be made into an ordered field because $1 = 1^2$ would have to be positive, but $1 + 1 + 1 = 0$.

2. F will not be a field because we will have $2 \cdot 2 = 0$.

3. As in Problem 1, conditions (2), (3), (5), and (6) are clear. Condition (1) can be checked case-by-case. To check (5) we can assume that $a, b, c \neq 0, 1$. This leaves only the cases $(\alpha \cdot \beta) \cdot \alpha = (\alpha \cdot \beta) \cdot \alpha$ and $(\alpha \cdot \beta) \cdot \beta = \alpha \cdot (\beta \cdot \beta)$.

4. (a) $a + a = a\cdot(1 + 1) = a\cdot 0 = 0.$

(b) $0 = a^{-1}\cdot 0 = a^{-1}(a + a) = 1 + 1.$

5. (a) The assertion is obvious when $n = 1$. Suppose it is true for $n$. Then

$$\underbrace{(1 + \dots + 1)}_{m\text{ times}}\cdot\underbrace{(1 + \dots + 1)}_{(n + 1)\text{ times}} = \underbrace{(1 + \dots + 1)}_{m\text{ times}}\cdot(\underbrace{[1 + \dots + 1]}_{n\text{ times}} + 1)$$

$$= \underbrace{(1 + \dots + 1)}_{m\text{ times}}\cdot\underbrace{(1 + \dots + 1)}_{n\text{ times}} + \underbrace{(1 + \dots + 1)}_{m\text{ times}}$$

$$= \underbrace{(1 + \dots + 1)}_{mn\text{ times}} + \underbrace{(1 + \dots + 1)}_{m\text{ times}} = \underbrace{(1 + \dots + 1)}_{mn + m = m(n + 1)\text{ times}}.$$

(b) If $m$ were not prime, so that $m = k\ell$ for some $k, \ell < m$, then

$$0 = \underbrace{(1 + \dots + 1)}_{m=k\ell\text{ times}} = \underbrace{(1 + \dots + 1)}_{k\text{ times}}\cdot\underbrace{(1 + \dots + 1)}_{\ell\text{ times}} = a\cdot b.$$

Therefore either $a = 0$ or $b = 0$, contradicting the assumption about $m$.

6. (a) If this were not true, then $F$ would have infinitely many distinct elements, namely, those of the form

$$\underbrace{1 + \ldots + 1}_{n \text{ times}}$$

for all n.

(b) Suppose $m > n$. Then

$$\underbrace{1 + \ldots + 1}_{m-n \text{ times}} = (\underbrace{1 + \ldots + 1}_{m \text{ times}}) - (\underbrace{1 + \ldots + 1}_{n \text{ times}}) = 0.$$

7. The solutions are

$$x = \frac{\alpha d - \beta b}{ad - bc},$$

$$y = \frac{\alpha c - \beta a}{bc - ad}.$$

8. (a) One, namely 0.

(b) Of course, a might not have any square roots at all (for example, if $F = \mathbf{R}$ and $a = -1$). But if a has one square root b, then it also has the square root $-b$; moreover, if $c^2 = a = b^2$, then $(c - b)(c + b) = 0$, so $c = b$ or $c = -b$. Consequently b and $-b$ are the only square roots; these are distinct precisely when $1 + 1 \neq 0$.

9. (a) is a straightforward check.

(b) In part (a), the symbol 2 means 1 + 1, which is 0 in $F_2$; the solution in part (a) is correct only if $1 + 1 \neq 0$.

10. (a) Most conditions require only a straightforward check. To verify 5(ii), note that if $(x, y) \neq 0$, so that $x \neq 0$ and $y \neq 0$, then $x^2 - ay^2 \neq 0$, since a does not have a square root. Then

$$(x, y) \odot \left(\frac{x}{x^2 - ay^2}, \frac{-y}{x^2 - ay^2}\right) = (1, 0).$$

(b) is a straightforward check.

(c) (0, 1) is a square root of a.

11. (a) The inverse of $(a_1, a_2, a_3, a_4)$ is

$$\left(\frac{a_1}{\gamma}, -\frac{a_2}{\gamma}, -\frac{a_3}{\gamma}, -\frac{a_4}{\gamma}\right),$$

where $\gamma = a_1{}^2 + a_2{}^2 + a_3{}^2 + a_4{}^2$.

(b) Each entry in the following table is the product a·b where a is on the left and b is above.

| | i | j | k |
|---|---|---|---|
| i | -1 | k | -j |
| j | -k | -1 | i |
| k | j | -i | -1 |

[If we denote 1, i, j, and k by $v_1$, $v_2$, $v_3$, $v_4$, then the definition of multiplication can be written

$$\left(\sum_{i=1}^{4} a_i v_i\right)\cdot\left(\sum_{j=1}^{4} b_j v_j\right) = \sum_{i,j=1}^{4} a_i b_j (v_i v_j).$$

This allows a simpler proof that multiplication is associative, by first checking that it is associative for ± 1, ± i, ± j, ± k.]

# CHAPTER 29

1. (a) $f(0) = f(0 + 0) = f(0) + f(0)$, so $f(0) = 0$. Since f is an isomorphism and $0 \neq 1$, it follows that $f(1) \neq 0$. Consequently, the equation $f(1) = f(1 \cdot 1) = f(1) \cdot f(1)$ implies that $f(1) = 1$.

   (b) $0 = f(0) = f(a + -a) = f(a) + f(-a)$, so $f(-a) = -f(a)$. Similarly, $1 = f(1) = f(a \cdot a^{-1}) = f(a) \cdot f(a^{-1})$, so $f(a^{-1}) = f(a)^{-1}$.

2. As an example, a proof for (a) will be given. If $\alpha^2 + 1 = 0$ for some $\alpha$ in $F_1$, then by Problem 1, $0 = f(0) = f(\alpha^2 + 1) = f(\alpha \cdot \alpha) + f(1) = f(\alpha)^2 + 1$, so $f(\alpha)$ is a solution of the equation $x^2 + 1 = 0$ in $F_2$.

3. (1) If $x \neq y$, then $f(x) \neq f(y)$, so $g(f(x)) \neq g(f(y))$, so $(g \circ f)(x) \neq (g \circ f)(y)$.

   (2) If z is in $F_3$, then $z = g(y)$ for some y in $F_2$, and $y = f(x)$ for some x in $F_1$. Then $z = (g \circ f)(x)$.

   (3)

$$\begin{aligned}(g \circ f)(x + y) &= g(f(x + y)) = g(f(x) + f(y)) = g(f(x)) + g(f(y)) \\ &= (g \circ f)(x) + (g \circ f)(y), \\ (g \circ f)(x \cdot y) &= g(f(x \cdot y)) = g(f(x) \cdot f(y)) = g(f(x)) \cdot g(f(y)) \\ &= (g \circ f)(x) \cdot (g \circ f)(y).\end{aligned}$$

   (4) If $x < y$, then $f(x) < f(y)$, so $g(f(x)) < g(f(y))$, i.e., $(g \circ f)(x) < (g \circ f)(y)$.

4. $g^{-1} \circ f$ is an isomorphism from **R** to **R**, so $g^{-1} \circ f = I$, so $g = f$.

5. Let $f(x + iy) = x - iy$. [Since $i^2 = -1$ we must have $f(i)^2 = f(-1) = -1$, so $f(i) = i$ or $-i$, which suggests the answer. This particular isomorphism is the only one, aside from the identity, which any one can write down, but there are actually infinitely many others. This is one of those facts which requires, aside from a knowledge of algebra, some of the sophisticated theorems from set theory which will be found in references [8] and [9] of the Suggested Reading.]

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